Lecture 1 - Review of Homological Algebra

Note Title 1/15/2009

Graded modules:

Def a ($\mathbb{Z}$-) graded R-module is a sequence of R-modules

$$M_\bullet = \ldots M_{-2} \quad M_{-1} \quad M_0 \quad M_1 \quad M_2 \ldots$$

Can grade by any set. When talking about products, we index by a monoid.

Associated to a graded R-module $M_\bullet$, have an "underlying" R-module also called $M_\bullet$:

$$M_\bullet = \bigoplus_{n \in \mathbb{Z}} M_n .$$

Elements of $M_n \subseteq M_\bullet$ are homogeneous of degree n.

We follow Moore's convention: never consider sums of unequal degree.

Ex: $M_\bullet = \quad \ldots \quad 0 \quad \underset{0}{\mathbb{Z}} \quad \underset{1}{\mathbb{Z}} \quad \underset{2}{\mathbb{Z}} \quad \ldots$

This is the same as considering $\mathbb{Z}[x]$ with the "degree" grading.

Have a tensor product and an internal Hom:

Def $(M_\bullet \otimes N_\bullet)_k = \bigoplus_{p+q=k} M_p \otimes N_q$

Ex: $M_\bullet = \mathbb{Z}[x]$, $N_\bullet = \mathbb{Z}[y]$. Then $M_p = \mathbb{Z} \cdot x^p$, $N_q = \mathbb{Z} \cdot y^q$, so

$\bigoplus_{p+q=k} M_p \otimes N_q$ is the free $\mathbb{Z}$-module on $x^p y^{k-p}$, $0 \le p \le k$

$\leftrightarrow$ $M_\bullet \otimes N_\bullet = \mathbb{Z}[x,y]$ w/ degree grading. So this seems like the right notion.

Def $\mathrm{Hom}^k(M_\bullet, N_\bullet) = \prod_p \mathrm{Hom}_R(M_p, N_{p+k})$

So an element of $\mathrm{Hom}^k(M_\bullet, N_\bullet)$ is a sequence of homomorphisms, each of which raises degree by k.

$\otimes$ and Hom will let us define graded algebras:

Def: A graded R-algebra is a graded R-module $M_\bullet$, together with a degree 0 map $M_\bullet \otimes M_\bullet \xrightarrow{\mu} M_\bullet$

Since $\oplus$ is the categorical coproduct, μ is the same thing as a collection of maps $M_a \otimes_R M_b \longrightarrow M_{a+b}$ for all a,b.

We will write $\mu(m \otimes n)$ as $m \cdot n$.

Classical notions like associativity & unit are the same.

Def A graded algebra is *graded commutative* if

$$a \cdot b = (-1)^{\deg a \cdot \deg b} \, b \cdot a$$

So odd degree classes anticommute, while even things commute with everything.

Why this? Cohomology of spaces is graded commutative.

Differential Graded Modules & Homology

Def A *differential graded module* is a pair $(M_\bullet, d)$ where $M_\bullet$ is a graded module, $d \in \operatorname{Hom}^{\pm 1}(M_\bullet, M_\bullet)$ &

$$d \circ d = 0.$$

If d has degree -1, say it is *homological*

If d has degree $+1$, say it is *cohomological*

So a dgm is a sequence of R-modules

$$\cdots \to M_{-2} \xrightarrow{d_{-2}} M_{-1} \xrightarrow{d_{-1}} M_0 \xrightarrow{d_0} M_1 \xrightarrow{d_1} M_2 \to \cdots$$

with a map between adjacent ones, the two-fold composites of which are zero. $\Rightarrow$ dgm $\longleftrightarrow$ chain complex.

Def The *cycles*, $Z(M)$, are the kernel of d

The *boundaries*, $B(M)$, are the image of d

Since $d^2 = 0$, $B(M) \subseteq Z(M)$, and can take the quotient.

Def The *homology* of M, $H(M)$, is $Z(M)/B(M)$

Z, B, & H are graded: $Z_k = \{ m \in M_k \mid d(m) = 0 \} = Z \cap M_k$

$B_k = \{ m \in M_k \mid \exists n \in M_{k-1} \text{ s.t. } d(n) = m \}$

And $H_k(M) = Z_k(M)/B_k(M)$.

We'll sometimes call an element n s.t. $d(n) = m$ a <u>null-homotopy</u> of m. Sometimes also a <u>null-bordism</u>.

<u>Ex</u>
1. Singular (co)homology
2. Cellular (co) homology
3. Simplicial abelian groups.

<u>Ex</u>: $0 \to \underset{0}{\mathbb{Z}} \xrightarrow{0} \underset{1}{\mathbb{Z}} \xrightarrow{p} \underset{2}{\mathbb{Z}} \xrightarrow{0} \underset{3}{\mathbb{Z}} \xrightarrow{p} \cdots$ has

$$\begin{array}{lccccccccccc}
Z(M): & 0 & \to & \mathbb{Z} & \to & 0 & \to & \mathbb{Z} & \to & 0 & \to & \cdots \\
 & \cup & & \cup & & \cup & & \cup & & \cup & & \\
B(M): & 0 & \to & 0 & \to & 0 & \to & p\mathbb{Z} & \to & 0 & \to & \cdots \\
H(M): & 0 & & \mathbb{Z} & & 0 & & \mathbb{Z}/p & & 0 & & \cdots
\end{array}$$

so

$$: \quad H_k(M) = \begin{cases} 0 & k<0 \text{ or } k \text{ odd} \\ \mathbb{Z}/p & k>0,\ k \text{ even} \\ \mathbb{Z} & k=0 \end{cases}$$

Have maps of dgms:

<u>Def</u> Let $(M_\bullet, d_M)$, $(N_\bullet, d_N)$ be dgm. A <u>homomorphism</u> is an element $f \in \mathrm{Hom}^0(M_\bullet, N_\bullet)$ s.t. $f(d_M(m)) = d_N(f(m))$ $\forall m$.

<u>Prop</u> A map of dgms induces a homomorphism of homology.

<u>Ex</u>

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & \mathbb{Z} \arrow[r, "0"] \arrow[d] & \mathbb{Z} \arrow[r, "p"] \arrow[d] & \mathbb{Z} \arrow[r, "0"] \arrow[d] & \mathbb{Z} \arrow[r, "p"] \arrow[d] & \mathbb{Z} \arrow[r, "0"] \arrow[d] & \mathbb{Z} \arrow[r] \arrow[d] & \cdots \\
0 \arrow[r] & \mathbb{Z}/p \arrow[r, "0"] & \mathbb{Z}/p \arrow[r, "0"] & \mathbb{Z}/p \arrow[r, "0"] & \mathbb{Z}/p \arrow[r, "0"] & \mathbb{Z}/p \arrow[r, "0"] & \mathbb{Z}/p \arrow[r] &
\end{tikzcd}$$

The homology of the bottom row is $\begin{cases} \mathbb{Z}/p & * \geq 0 \\ 0 & * < 0 \end{cases}$

& $H_*(\text{top}) \longrightarrow H_*(\text{bottom})$ is the obvious map.

So even though the map from the top to bottom is surjective, the map in homology is not!

<u>Aside</u> / <u>homework</u> If $(M_\bullet, d_M)$ and $(N_\bullet, d_N)$ are dgms, then $(M_\bullet \otimes N_\bullet, d)$, where $d(m \otimes n) = d_M(m) \otimes n + (-1)^{\deg m} m \otimes d_N(n)$, is a dgm ?

$(\mathrm{Hom}^\bullet(M_\bullet, N_\bullet), d)$ where

$$(df)(m) = d_N(f(m)) + (-1)^{\deg f + 1} f(d_M(m))$$

is a dgm. Find $Z_0(\mathrm{Hom}(M_\bullet, N_\bullet))$ and B_0.

<u>Def</u> A <u>differential graded algebra</u> is a dgm & a graded algebra where

Leibnitz Rule: $$d(m \cdot n) = d(m)\cdot n + (-1)^{\deg m} m \cdot d(n).$$

<u>Prop</u> A dga is a dgm with a map of dgms:

$$M_\bullet \otimes M_\bullet \longrightarrow M_\bullet$$

HW: The Leibnitz rule ensures that the multiplication map μ induces a multiplication on $H_*(M)$.

<u>Ex</u> The singular cochain complex is a dga. It is <u>not</u> comm!

Homology Long Exact Sequence

<u>Def</u> A d.g.m $(M_\bullet, d)$ is <u>exact</u> if $\ker(d) = \mathrm{Im}(d)$.

Also call this <u>acyclic</u>

So exact $\longleftrightarrow H_*(M) = 0$.

<u>Def</u> $0 \to K_\bullet \xrightarrow{i} M_\bullet \xrightarrow{p} C_\bullet \to 0$ is a <u>short exact sequence</u> (of dgms, etc) if

① i is injective.

② $\ker(p) = \mathrm{Im}(i)$

③ p is surjective.

HW/Thm If $0 \to K_\bullet \to M_\bullet \to C_\bullet \to 0$ is a SES of dgm, then there is an exact sequence

$$\to H_n(K) \xrightarrow{i} H_n(M) \xrightarrow{p} H_n(C) \xrightarrow{\partial} H_{n-1}(K) \to \cdots$$

∂ is the <u>connecting homomorphism</u> & the degree is that of d in K, M, C.

Ex

$$
\begin{tikzcd}
3 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] \\
2 & \mathbb{Z} \arrow[r, "\cdot 2"] \arrow[d, "\cdot 2"] & \mathbb{Z} \arrow[r] \arrow[d, "\cdot 2"] & \mathbb{Z}/2 \arrow[d, "0"] \\
1 & \mathbb{Z} \arrow[r, "\cdot 2"'] \arrow[d] & \mathbb{Z} \arrow[r] \arrow[d] & \mathbb{Z}/2 \arrow[d] \\
0 & 0 & 0 & 0 \\
 & K_\bullet & M_\bullet & C_\bullet
\end{tikzcd}
$$

$$H_*(K) = H_*(M) = \begin{cases} \mathbb{Z}/2 & * = 1 \\ 0 & * \neq 1 \end{cases}$$

$$H_*(C) = \begin{cases} \mathbb{Z}/2 & * = 1,2 \\ 0 & * \neq 1,2 \end{cases}$$

Consider 1 in C_2. This generates H_2. Since $M_2 \to C_2$ is onto, choose a lift $1+2k$ of 1. Different lifts differ by the image of K_2 in M_2. Apply the boundary d in M: $1+2k \longmapsto 2+4k \in M_1$. This maps to zero in C_1 ($1+2k$ is a lift of a cycle) So it is in the image of K_1 in M_1. Pull back to K_1, getting $1+2k$. This is in $Z_1(K)$, and passing to $H_1(K)$ gives the nontrivial element in $H_1(K)$.

Note that the different lifts give things that differ by a boundary. So the cycle is not uniquely defined, but the homology class is!

The LES:

$$
\begin{array}{ccccccccccccccc}
\cdots \longrightarrow & 0 & \to & 0 & \longrightarrow & \mathbb{Z}/2 & \xrightarrow{\cong} & \mathbb{Z}/2 & \xrightarrow{0} & \mathbb{Z}/2 & \xrightarrow{\cong} & \mathbb{Z}/2 & \to & 0 & \to \cdots \\
 & H_2K & & H_2M & & H_2C & & H_1K & & H_1M & & H_1C & & &
\end{array}
$$

just constructed