Lecture 9 – Tangent Planes

Note Title

Start with a natural families of curves and tangent vectors.

Given a point (a,b):

let C_1 be the intersection of $z=f(x,y)$ with $y=b$ $\&$

let C_2 be the intersection of $z=f(x,y)$ with $x=a$.

x $\&$ y themselves give natural parametrizations of these curves:

$\bar{r}_1(x) = \langle x, b, f(x,b)\rangle = \langle x, b, g(x)\rangle$ where $g(x)=f(x,b)$.

$\bar{r}_2(y) = \langle a, y, f(a,y)\rangle = \langle a, y, h(y)\rangle$ where $h(y)=f(a,y)$.

@ (a,b), have

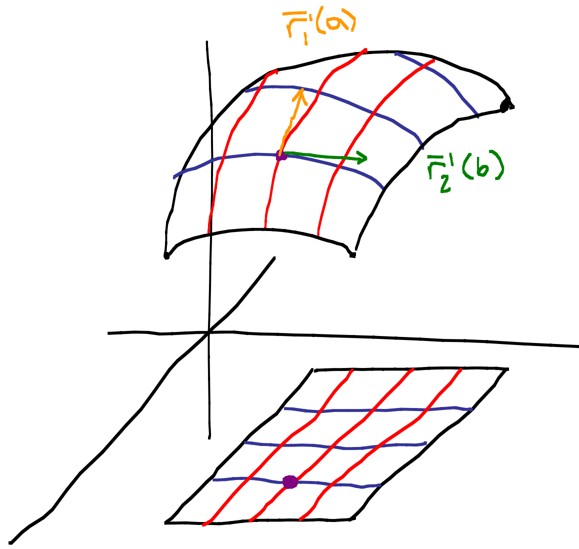
$$\bar{r}_1'(a) = \langle 1, 0, g'(a)\rangle \quad \&$$

$$\bar{r}_2'(b) = \langle 0, 1, h'(b)\rangle$$

What are $g'(a)$ $\&$ $h'(b)$? The partials!

$$\bar{r}_1'(a) = \left\langle 1, 0, \frac{\partial f}{\partial x}(a,b)\right\rangle$$

$$\bar{r}_2'(b) = \left\langle 0, 1, \frac{\partial f}{\partial y}(a,b)\right\rangle$$



These are 2 tangent vectors to the surface. These determine a plane:

$\bar{r}_0 = \langle a, b, f(a,b)\rangle$

$$\bar{n} = \bar{r}_1'(a) \times \bar{r}_2'(b) = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ 1 & 0 & \frac{\partial f}{\partial x}(a,b) \\ 0 & 1 & \frac{\partial f}{\partial y}(a,b) \end{vmatrix} = \left\langle -\frac{\partial f}{\partial x}(a,b), -\frac{\partial f}{\partial y}(a,b), 1\right\rangle$$

<u>Tangent Plane</u>:

$$\bar{n}\cdot(\bar{r}-\bar{r}_0) = 0 \iff -\frac{\partial f}{\partial x}(a,b)(x-a) - \frac{\partial f}{\partial y}(a,b)(y-b) + (z-f(a,b)) = 0$$

$$\iff z - f(a,b) = \frac{\partial f}{\partial x}(a,b)(x-a) + \frac{\partial f}{\partial y}(a,b)(y-b)$$

This is also the linear approximation to $f(x,y)$ at (a,b):

$$L(x,y) = f(a,b) + \frac{\partial f}{\partial x}(a,b)\cdot(x-a) + \frac{\partial f}{\partial y}(a,b)\cdot(y-b)$$

This should be a better than first order approx to f.

<u>Ex</u>: $f(x,y) = x^2 + 2y^2$ @ $(2,1)$

$$\left.\begin{aligned} &f(2,1) = (2)^2 + 2\cdot(1)^2 = 6 \\ &f_x(x,y) = 2x \Rightarrow f_x(2,1) = 4 \\ &f_y(x,y) = 4y \Rightarrow f_y(2,1) = 4 \end{aligned}\right\} = L(x,y) = 6 + 4(x-2) + 4(x-1)$$

So $\underset{\underset{8.1875}{\parallel}}{f(2\tfrac{1}{4}, 1\tfrac{1}{4})} \approx L(2\tfrac{1}{4}, \tfrac{1}{4}) = 6 + 4(\tfrac{1}{4}) + 4(\tfrac{1}{4}) = 8$

This is pretty close.

Say that f is differentiable at (a,b) if $L(x,y)$ is a better than 1^{st} order approx to f:

$$f(x,y) - L(x,y) = \epsilon_1(x,y)\cdot(x-a) + \epsilon_2(x,y)\cdot(y-b)$$

where $\lim\limits_{(x,y)\to(a,b)} \epsilon_i(x,y) = 0$.

<u>Thm</u>: If f_x & f_y exist near (a,b) & are continuous there, then f is differentiable.

So most functions we encounter are differentiable.

Summerizing:

partial derivatives $\Longrightarrow$ Tangent Plane

$\updownarrow$

linear approximation

$\updownarrow$

differentials.

<u>Differentials</u> $f(x,y) - f(a,b)$ is defined to be Δz.

$x - a$ " " " " Δx

$y - b$ " " " " Δy

Then if f is differentiable, then

$$\Delta z \approx \frac{\partial f}{\partial x}(a,b)\cdot\Delta x + \frac{\partial f}{\partial y}(a,b)\cdot\Delta y$$

This gets better and better as $(\Delta x, \Delta y) \to (0,0)$.

The change in the tangent plane $= L(x,y) - f(a,b)$

is the differential dz. So as $(\Delta x, \Delta y) \to (0,0)$,

$$dz = \Delta z.$$

Ex $f(x,y) = 2x^3 + y^3$ at $(1,2)$

$$\left.\begin{aligned} f(1,2) &= 10 \\ \frac{\partial f}{\partial x}(1,2) &= 6 \\ \frac{\partial f}{\partial y}(1,2) &= 12 \end{aligned}\right\} \Rightarrow \begin{aligned} L(x,y) &= 10 + 6(x-1) + 12(y-2) \\ dz &= 6\,dx + 12\,dy \end{aligned}$$

& Can approx $f(1\tfrac{1}{6}, 1\tfrac{5}{6}) = 9.33$

$$L(1\tfrac{1}{6}, 1\tfrac{5}{6}) = 10 + 6\cdot(\tfrac{1}{6}) + 12(-\tfrac{1}{6}) = 9$$

In this case, $f(x,y) = \underbrace{10 + 6(x-1) + 12(y-2)}_{L(x,y)} + \underbrace{6(x-1)^2 + 6(y-2)^2 + (x-1)^3 + (y-2)^3}_{\text{error but computable}}$

So $\epsilon_1(x,y) = 6(x-1) + (x-1)^2$

$\epsilon_2(x,y) = 6(y-2) + (y-2)^2$