Lecture 7 - Functions of 2 Variables

Note Title

<u>Def</u> A function of 2 variables is a rule that assigns to each pair of numbers (x,y) (taken from a subset D of $\mathbb{R}^2$) a real number $f(x,y)$. The set D of all possible values of (x,y) is the domain, while the set of all possible values of $f(x,y)$ is the image.

<u>Ex</u>: •) $f(x,y) = x^2 + y^2$ is a function of two variables. The domain is $\mathbb{R}^2$, while the image is the non-negative real numbers.

•) The temperature of a point in the US is a function of 2 var (say latitude & longitude).

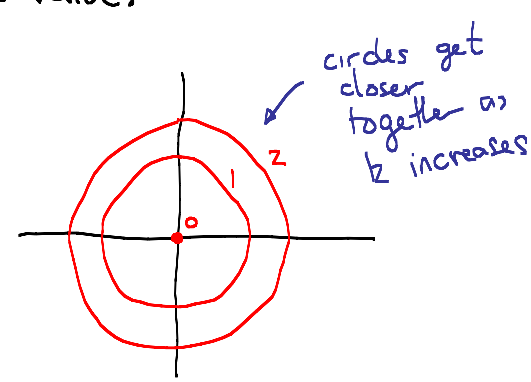
•) The elevation of a point in the US is a function of 2 var.

The equation $z = f(x,y)$ defines a surface in $\mathbb{R}^3$, the graph of $f(x,y)$.

We can understand f's graph through level curves (= z-traces): plot on one set of axes the curves $f(x,y) = k$ for various values of k & label the curves w/ their k-value.

<u>Ex</u>: •) $f(x,y) = x^2 + y^2$

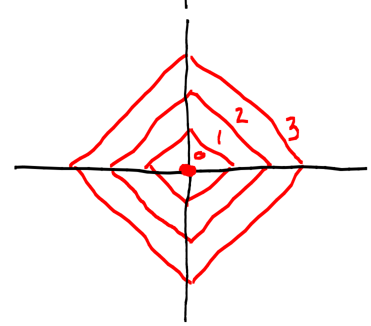
$$x^2+y^2 = k \quad \text{is} \quad \begin{cases} \text{a circle} & k>0 \\ \text{a point} & k=0 \\ \text{nothing} & k<0 \end{cases}$$



The graph is a paraboloid

•) $f(x,y) = |x| + |y|$

$$|x|+|y| = k \quad \text{is} \quad \begin{cases} \text{a diamond} & k>0 \\ \text{a point} & k=0 \\ \text{nothing} & k<0 \end{cases}$$



The graph is a "square cone".

•) $f(x,y)$ = height at (x,y). Then the sketch of level curves for some fixed increment of k gives a topographical map. The level curves are therefore sometimes called isoclines.

Limits and continuity behave differently.

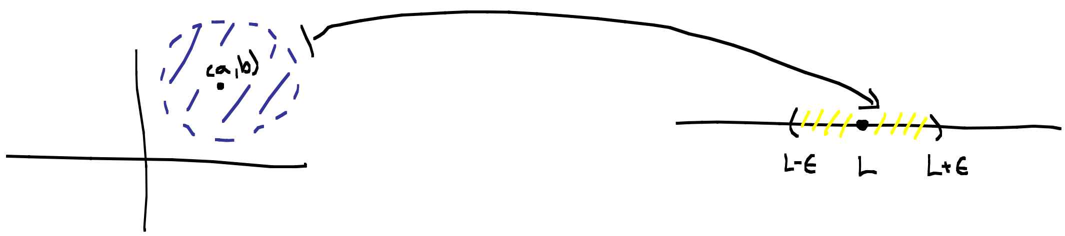
<u>Def</u> We say that "the limit as (x,y) approaches (a,b) of f is L", written

$$\lim_{(x,y)\to(a,b)} f(x,y) = L$$

if for every $\epsilon>0$, we can find a $\delta>0$ s.t.

$$0< d((x,y),(a,b)) < \delta \Rightarrow |f(x,y)-L| < \epsilon.$$

In other words, $\lim_{(x,y)\to(a,b)} f(x,y)=L$ if for any small interval around L we can find a small disk about (a,b) that maps entirely into that interval.



<u>Def</u> f is continuous at (a,b) if

$$\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b).$$

Limits can be hard to evaluate.

<u>Ex</u> $f(x,y) = x$. $\lim_{(x,y)\to(0,0)} f(x,y) = 0$.

Given $\epsilon>0$, we must produce δ s.t.

$$0 < d((x,y),(0,0)) < \delta \Rightarrow |x-0| < \epsilon$$

Since $|x| = \sqrt{x^2} \le \sqrt{x^2+y^2}$ for all x,y, if $\delta=\epsilon$, then

$$|x| \le \sqrt{x^2+y^2} = d((x,y),(0,0)) < \delta = \epsilon. \checkmark$$

Ex $\lim_{(x,y)\to(0,0)} x^2+y^2 = 0$

Given $\epsilon > 0$, we must produce $\delta > 0$ s.t.

$$0 < d((x,y),(0,0)) < \delta \Rightarrow |x^2+y^2-0| < \epsilon$$

Know that $d((x,y),(0,0)) = \sqrt{x^2+y^2} < \delta$. Squaring gives

$$x^2+y^2 < \delta^2.$$

So if we choose $\delta = \sqrt{\epsilon}$, then

$$\sqrt{x^2+y^2} < \delta \Rightarrow x^2+y^2 < \delta^2 = \epsilon.$$

Know several basic facts about limits and continuous functions.

1. If f, g, h are continuous functions of one or two variables, then any composites which make sense are continuous (ie $f(h(x,y))$, $h(f(x), g(y))$, $h(f(x), g(x,y))$, etc).
2. If f and g are continuous, then
   - $(f+g)(x,y) = f(x,y) + g(x,y)$ is,
   - $(f \cdot g)(x,y) = f(x,y) \cdot g(x,y)$ is,
   - and so long as $g(x,y) \neq 0$, $(f/g)(x,y) = f(x,y)/g(x,y)$ is.

So basically all the functions you can make from continuous 1-var functions will be continuous:

Ex
- $\sin(x+y^3+x^4)e^{-x^2-y^2}$
- xe^y
- $x/_{2+\cos y}$

How can we check if limits don't exist? Approach along different paths.

If $\lim_{(x,y)\to(a,b)} f(x,y)$ exists, then it is also the limit along any path to (a,b) (this is the analogue of left and right limits).

Ex: $f(x,y) = \dfrac{x^2}{x^2+y^2}$ & $(a,b) = (0,0)$. We'll approach along lines: $y = mx$

If the limit exists, it will be independent of our choice of path (ie ind of m).

$$\lim_{x \to 0} \frac{x^2}{x^2 + (mx^2)} = \lim_{x \to 0} \frac{\cancel{x^2} \cdot 1}{\cancel{x^2}(1+m^2)} = \lim_{x \to 0} \frac{1}{1+m^2} = \frac{1}{1+m^2}.$$

This is not independent of m, so the limit does not exist!