Lecture 6 – Arc Length

Note Title

2 goals:
- good geometric concept
- natural parameterization

I. Definitions:

differential: $d\bar{r} = \langle dx, dy, dz \rangle$ is the <u>linear</u> change in $\bar{r}$.

$$d\bar{r} = \langle f'(t)\,dt,\ g'(t)\,dt,\ h'(t)\,dt \rangle \qquad \text{vs.}$$

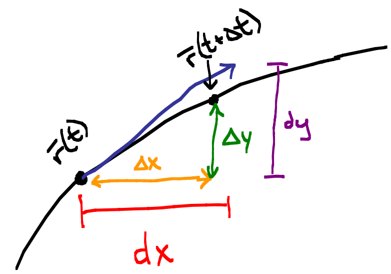
$$\Delta\bar{r} = \bar{r}(t+\Delta t) - \bar{r}(t)$$

<u>Thm</u> $\Delta t = dt$, and if Δt is very small,

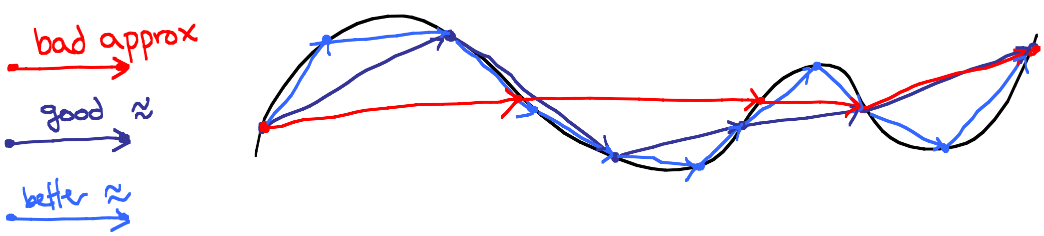
$$\Delta\bar{r} \approx d\bar{r}.$$

This is the basic fact of vector calculus.

$$\Delta x \approx dx \qquad \Delta z \approx dz$$
$$\Delta y \approx dy$$



Arc length: Approximate the curve with line segments:



This lets us approximate the length by the lengths of the vectors: $|\Delta\bar{r}(t_i)| \approx |d\bar{r}(t_i)|$

<u>Def</u> The arc length of the curve defined by $\bar{r}(t)$ between a & b is

$$s = \int_a^b \underbrace{\sqrt{\left(\frac{df}{dt}\right)^2 + \left(\frac{dg}{dt}\right)^2 + \left(\frac{dh}{dt}\right)^2}\ dt}_{ds = |d\bar{r}|}$$

<u>Ex</u> $\bar{r}(t) = \langle \cos t, \sin t, t \rangle$ then the length of one revolution is

$$s = \int_0^{2\pi} \sqrt{(-\sin t)^2 + (\cos t)^2 + 1}\, dt = \int_0^{2\pi} \sqrt{2}\, dt = 2\sqrt{2}\,\pi.$$

Ex: $\bar{r}(t) = \langle \ln(\cos t), t, 1 \rangle$

$r'(t) = \langle -\tan t, 1, 0 \rangle$

$$ds = |r'(t)|\, dt = \sqrt{\tan^2 t + 1}\, dt = \sec t\, dt$$

so arc length between 0 and $\pi/4$ is

$$\int_0^{\pi/4} \sec t\, dt = \ln(\sec t + \tan t)\Big|_0^{\pi/4} = \ln(\sqrt{2}+1).$$

We can parameterize with respect to arclength:

$$s = \int_a^t ds = \text{some (increasing) function of } t, \text{ so can solve for } t.$$

Ex: $\bar{r}(t) = \langle \sin 3t, \cos 3t, 4t \rangle$

$$\bar{r}'(t) = \langle 3\cos 3t, -3\sin 3t, 4 \rangle \Rightarrow ds = \sqrt{(3\cos 3t)^2 + (3\sin 3t)^2 + 16} = 5$$

$$\Rightarrow \quad s(t) = \int_0^t ds = \int_0^t 5\, dt = 5t \quad \Rightarrow \quad t = s/5 \quad \&$$

$$\bar{r}(s) = \left\langle \sin \tfrac{3}{5}s, \cos \tfrac{3}{5}s, \tfrac{4}{5}s \right\rangle$$

Really nice property:

$$T(s) = \frac{d\bar{r}}{ds}$$

chain rule

Why? $\bar{r}'(s) = \dfrac{d\bar{r}}{ds} = \dfrac{d\bar{r}}{dt} \cdot \dfrac{dt}{ds} = \dfrac{d\bar{r}/dt}{ds/dt} = \dfrac{d\bar{r}/dt}{|d\bar{r}/dt|} = \bar{T}.$

This makes s the most natural coordinate choice.

Since $\bar{T}\cdot\bar{T} = 1$, $\dfrac{d}{ds}(\bar{T}\cdot\bar{T}) = 0$

Product rule $\Rightarrow$ $2\dfrac{d\bar{T}}{ds}\cdot\bar{T} = 0 \Rightarrow \dfrac{d\bar{T}}{ds}$ is $\perp$ to $\bar{T}$.

Def The unit normal vector is defined by

$$\bar{N} = \frac{d\bar{T}/ds}{|d\bar{T}/ds|}$$

The length of $d\bar{T}/ds$ also gets a name.

<u>Def</u> The curvature of a curve is given by

$$\kappa = \left| d\bar{T}/ds \right|.$$

This measures how sharply the curve bends at a point

<u>Ex</u>: $\bar{r}(s) = \left\langle \sin \frac{3}{5}s,\ \cos \frac{3}{5}s,\ \frac{4}{5}s \right\rangle$

$$\bar{T}(s) = \left\langle \frac{3}{5}\cos \frac{3}{5}s,\ -\frac{3}{5}\sin \frac{3}{5}s,\ \frac{4}{5} \right\rangle$$

$$\bar{T}'(s) = \left\langle \frac{-9}{25}\sin \frac{3}{5}s,\ -\frac{9}{25}\cos \frac{3}{5}s,\ 0 \right\rangle = \underbrace{\frac{9}{25}}_{\kappa} \underbrace{\left\langle -\sin \frac{3}{5}s,\ -\cos \frac{3}{5}s,\ 0 \right\rangle}_{\bar{N}}$$

Even w/o using s, we can get κ and $\bar{N}$.

$$\frac{d\bar{T}}{ds} = \frac{d\bar{T}}{dt} \cdot \frac{dt}{ds} = \frac{d\bar{T}/dt}{ds/dt} = \frac{\bar{T}'(t)}{|\bar{r}'(t)|}.$$

so $\quad \kappa = \dfrac{|\bar{T}'(t)|}{|\bar{r}'(t)|} \quad$ & $\quad \bar{N} = \dfrac{\bar{T}'(t)}{|\bar{T}'(t)|}$

Given $\bar{T}$ & $\bar{N}$, we get a third vector.

<u>Def</u> The unit binormal vector is defined by

$$\bar{B} = \bar{T} \times \bar{N}.$$

Together, $\bar{T}, \bar{N}$, and $\bar{B}$ carry huge amounts of info about the geometry of the curve.

<u>Ex</u>: $r(t) = \langle \sin t,\ t,\ \cos t \rangle$

$|r'(t)| = \sqrt{2}$, & $t = s/\sqrt{2}$: $\bar{r}(s) = \left\langle \sin s/\sqrt{2},\ s/\sqrt{2},\ \cos s/\sqrt{2} \right\rangle$

$$\bar{T}(s) = \left\langle \frac{1}{\sqrt{2}}\cos s/\sqrt{2},\ 1/\sqrt{2},\ -\frac{1}{\sqrt{2}}\sin s/\sqrt{2} \right\rangle$$

$$\frac{d\bar{T}}{ds} = \left\langle -\frac{1}{2}\sin s/\sqrt{2},\ 0,\ -\frac{1}{2}\cos s/\sqrt{2} \right\rangle = \underbrace{\frac{1}{2}}_{\kappa} \underbrace{\left\langle -\sin s/\sqrt{2},\ 0,\ -\cos s/\sqrt{2} \right\rangle}_{\bar{N}}$$

$$\bar{B} = \bar{T} \times \bar{N} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ 1/\sqrt{2}\cos s/\sqrt{2} & 1/\sqrt{2} & -1/\sqrt{2}\sin s/\sqrt{2} \\ -\sin s/\sqrt{2} & 0 & -\cos s/\sqrt{2} \end{vmatrix} = \left\langle -\frac{1}{\sqrt{2}}\cos s/\sqrt{2},\ 1/\sqrt{2},\ 1/\sqrt{2}\sin s/\sqrt{2} \right\rangle$$