Lecture 5 - Vector Valued Functions

Note Title

Start doing calculus with vectors.

Let $\bar{r}$ be a function of $t \in \mathbb{R}$ such that for all t, $\bar{r}(t)$ is a vector. This is a "vector valued function".

In coordinates: $\bar{r}(t) = \langle f(t), g(t), h(t) \rangle = f(t) \cdot \bar{i} + g(t) \cdot \bar{j} + h(t) \cdot \bar{k}$.

<u>Ex</u> $\bar{r}(t) = \langle t^2, e^t, t \rangle$

$\bar{s}(t) = \langle 1-t, 3+t, 8t-1 \rangle$

Can do all our usual calculus things.

<u>Def</u> Let $\bar{r}(t) = \langle f(t), g(t), h(t) \rangle$. Then

$$\lim_{t \to a} \bar{r}(t) = \langle \lim_{t \to a} f(t), \lim_{t \to a} g(t), \lim_{t \to a} h(t) \rangle,$$ assuming all exist.

In other words: limits are coordinate-wise.

<u>Ex</u>: $\bar{r}(t) = \langle t^2+1, t-1, 8t \rangle$

$$\lim_{t \to 0} \bar{r}(t) = \langle \lim_{t \to 0} (t^2+1), \lim_{t \to 0} t-1, \lim_{t \to 0} 8t \rangle = \langle 1, -1, 0 \rangle$$

From limits, we get continuity.

<u>Def</u> $\bar{r}(t)$ is continuous at a if $\lim_{t \to a} \bar{r}(t) = \bar{r}(a)$.

Since limits and evaluation are coordinate-wise, this is the same thing as "all coordinate functions are continuous at a".

Most functions we encounter will be continuous.

Now an aside about sketching. As t varies, $\bar{r}(t)$ traces out an oriented curve. We can understand the curve abstractly by looking at surfaces it sits on.

Basic idea is to eliminate t from the <u>parametric form</u> of $\bar{r}(t)$:

$$x = f(t), \quad y = g(t), \quad z = h(t).$$

Ex: $\bar{r}(t) = \langle t, t^2, t^3 \rangle \longleftrightarrow$ $x = t$, $y = t^2$, $z = t^3$

can get 3 equations from these:

$y = x^2$

$z = x^3$

$y^3 = z^2$

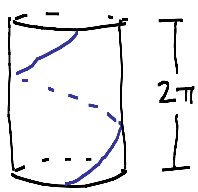
all give cylinders in space.

The curve sits on the intersection of all 3 cylinders.

Ex 2: $\bar{r}(t) = \langle \cos t, \sin t, t \rangle$

3 surfaces: $x = \cos z$, $y = \sin z$ — x & y oscillate as z changes

$x^2 + y^2 = 1$ — x & y constrained

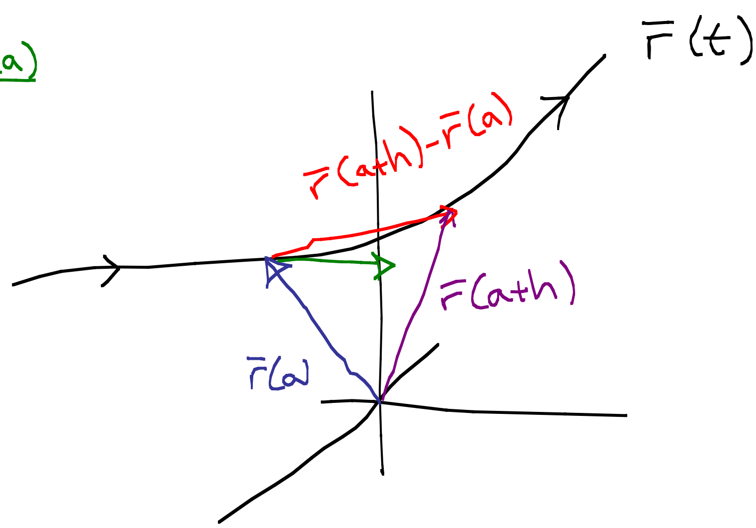


This is a helix.

The fact that t is present gives us an orientation = choice of direction.

From the tangent vector:

$$\bar{r}'(a) = \lim_{h \to 0} \frac{\bar{r}(a+h) - \bar{r}(a)}{h}$$



The picture suggests that the direction of $\bar{r}'(a)$ is the tangent direction at $\bar{r}(a)$, just as in ordinary calculus

Limits, scalar mult, & addition all coord-wise, so

$$\bar{r}'(t) = \langle f'(t), g'(t), h'(t) \rangle$$

Ex: $\bar{r}(t) = \langle \cos t, \sin t, t \rangle$

$\Rightarrow \bar{r}'(t) = \langle -\sin t, \cos t, 1 \rangle$

So $\bar{r}(0) = \langle 0, 1, 1 \rangle$

Often want just the direction.

Def If $\bar{r}'(t) \neq 0$, then the unit tangent vector at t is

$$T(t) = \frac{\bar{r}'(t)}{|\bar{r}'(t)|}.$$

$\bar{r}$ is smooth at t if $\bar{r}'(t)$ exists and is non-zero.

Ex: $\bar{r}$ as above,

$$\bar{r}'(t) = \langle -\sin t, \cos t, 1 \rangle \Rightarrow |\bar{r}'(t)| = \sqrt{\sin^2 t + \cos^2 t + 1} = \sqrt{2}$$

$$\Rightarrow T(t) = \left\langle -\tfrac{1}{\sqrt{2}} \sin t, \tfrac{1}{\sqrt{2}} \cos t, 1/\sqrt{2} \right\rangle$$

Ex: $\bar{r}(t) = \langle t^2, t^3, t^4 \rangle$ Then $\bar{r}$ is not smooth at zero:

$$\bar{r}'(t) = \langle 2t, 3t^2, 4t^3 \rangle \Rightarrow \bar{r}'(0) = \bar{0}.$$

Integration is also done coordinatewise:

$$\int_a^b \bar{r}(t)\, dt = \left\langle \int_a^b f(t)\, dt, \int_a^b g(t)\, dt, \int h(t)\, dt \right\rangle$$

Ex $\bar{r}(t) = \langle \cos t, \sin t, t \rangle$

$$\int \bar{r}(t)\, dt = \langle \sin t, -\cos t, t^2/2 \rangle + \bar{c}.$$