Lecture 3 –

Note Title

② Algebraic: $(\bar u+\bar v)\times\bar w = \bar u\times\bar w+\bar v\times\bar w$ & $\bar u\times(\bar v+\bar w) = \bar u\times\bar v+\bar u\times\bar w$

So $\langle a,b,c\rangle\times\langle x,y,z\rangle = (a\bar\imath+b\bar\jmath+c\bar k)\times(x\bar\imath+y\bar\jmath+z\bar k)$

$= ax\,\cancel{\bar\imath\times\bar\imath} + ay\,\bar\imath\times\bar\jmath + az\,\bar\imath\times\bar k + bx\,\bar\jmath\times\bar\imath + by\,\cancel{\bar\jmath\times\bar\jmath} + bz\,\bar\jmath\times\bar k + cx\,\bar k\times\bar\imath + cy\,\bar k\times\bar\jmath + cz\,\cancel{\bar k\times\bar k} =$

$(bz-cy)\bar\imath + (cx-az)\bar\jmath + (ay-bx)\bar k$

So $\langle 1,2,0\rangle\times\langle 2,1,0\rangle =$

$(2\cdot 0-0\cdot 1)\bar\imath + (0\cdot 2-1\cdot 0)\bar\jmath + (1\cdot 1-2\cdot 2)\bar k = -3\bar k$ ✓

Last remark: Can also remember via a "determinant"

$$\bar u\times\bar v = \begin{vmatrix} \bar\imath & \bar\jmath & \bar k \\ a & b & c \\ x & y & z \end{vmatrix}$$

How do we evaluate this? Repeat the 1st 2 columns & multiply out diagonals:

$$\left|\begin{array}{ccc|cc} \bar\imath & \bar\jmath & \bar k & \bar\imath & \bar\jmath \\ a & b & c & a & b \\ x & y & z & x & y \end{array}\right|$$

Down & right gets a +

Up & right gets a −

$= (bz\,\bar\imath + cx\,\bar\jmath + ay\,\bar k) - (bx\,\bar k + cy\,\bar\imath + az\,\bar\jmath)$

$= (bz-cy)\bar\imath + (cx-az)\bar\jmath + (ay-bx)\bar k$

Ex: $\bar u = \langle 1,3,5\rangle$

$\bar v = \langle 2,1,0\rangle$

$$\bar u\times\bar v = \left|\begin{array}{ccc|cc} \bar\imath & \bar\jmath & \bar k & \bar\imath & \bar\jmath \\ 1 & 3 & 5 & 1 & 3 \\ 2 & 1 & 0 & 2 & 1 \end{array}\right|$$

$(0\cdot 3\,\bar\imath + 2\cdot 5\,\bar\jmath + 1\cdot 1\,\bar k) -$

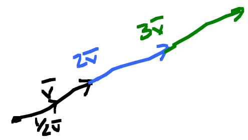
$(2\cdot 3\,\bar k + 1\cdot 5\,\bar\imath + 0\cdot 1\,\bar\jmath)$

$= -5\bar\imath + 10\bar\jmath - 5\bar k$

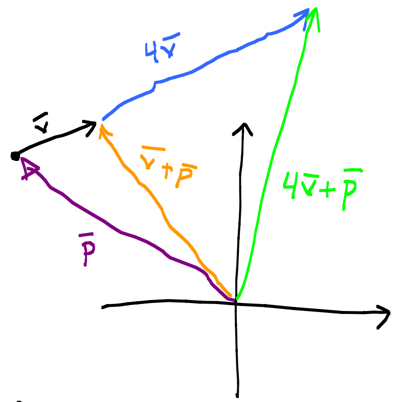
Lines: Already seen the basic concept:

If we fix the tail of a vector $\bar{v}$, then the tips of scalar multiples carve out a line:

Scalar mults: $t \cdot \bar{v}$



Now put in coord:



So the parametric equation of the line through (a,b,c) and parallel to $\bar{v} = \langle \alpha, \beta, \gamma \rangle$ is

$$x = \alpha t + a$$
$$y = \beta t + b$$
$$z = \gamma t + c$$

"parameter"

Ex: Line through $(1,2,3)$ and $\parallel$ to $\langle 3,0,1 \rangle$:

$$x = 3t+1$$
$$y = 2$$
$$z = t+3$$

Easiest to remember the vectors, not the points:

$$\bar{r}(t) = t \cdot \bar{v} + \bar{p}$$

This form depends on choice of $\bar{v}$ and $\bar{p}$ (ie $\bar{w} = 2\bar{v}$ and $\bar{q} = \bar{p} + \bar{v}$ gives a diff form)

Symmetric form: solve for t:

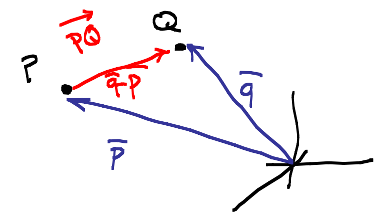
$$t = \frac{x-a}{\alpha}$$
$$= \frac{y-b}{\beta}$$
$$= \frac{z-c}{\gamma}$$

If one of $\alpha, \beta, \gamma = 0$, then just ignore that.

Ex $(t=)$ $\frac{x-1}{3} = z-3$, $y = 2$ is the symmetric form of line above.

Ex: "2 points determine a line" $P=(1,3,8)$, $Q=(2,1,-1)$. Find the equation of the line between P and Q.

Need $\bar{p}$ and $\bar{v}$. $\bar{p}$ is easy: $\bar{p}=\langle 1,3,8\rangle$



For $\bar{v}$: P & Q determine a vector $\overrightarrow{PQ}$:

$$\langle 2,1,-1\rangle-\langle 1,3,8\rangle=\langle 1,-2,-9\rangle$$

This is $\bar{v}$. So

$$\bar{r}(t)=\langle 1,3,8\rangle+t\langle 1,-2,-9\rangle$$
$$=\langle 1+t,\ 3-2t,\ 8-9t\rangle$$

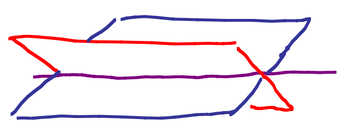
Q: What does the symmetric form tell us?

$$\frac{x-a}{\alpha}=\frac{y-b}{\beta}=\frac{z-c}{\gamma}$$

If we consider these pairwise, then we get lines in the corresponding plane (eg $\frac{x-a}{\alpha}=\frac{y-b}{\beta}$ gives a line in the (x,y)-plane)

$\Rightarrow$ planes in space.

The symmetric form describes the line as the intersection of planes.



(Idea: space is 3-d. each equation cuts that down by 1)

Equations of planes

A plane sees 2 of the 3 possible directions. There is one direction $\perp$ to the plane: the normal direction: $\bar{n}$

Plane determined by requiring that every vector from one point on the plane to another is $\perp$ to $\bar{n}$. Let $\bar{r}_0$ point to a point on the plane. Then

$$(x,y,z) \text{ on the plane } \longleftrightarrow \bar{n}\cdot(\bar{r}-\bar{r}_0)=0$$

Ex: Find the eqn of the plane through $(1,0,-2)$ and $\perp$ to $\langle 1,-1,6\rangle$

"$\perp$ to $\langle 1,-1,6\rangle$" $\longleftrightarrow$ $\bar{n} = \langle 1,-1,6\rangle$

$(1,0,-2)$ on plane $\Rightarrow$ $\bar{r}_0 = \langle 1,0,-2\rangle$, so

eqn of the plane is

$$\bar{n}\cdot(\bar{r}-\bar{r}_0) = \langle 1,-1,6\rangle \cdot \langle x-1, y, z+2\rangle = 0$$

$$\langle 1,-1,6\rangle \cdot \langle x-1, y, z+2\rangle = (x-1) - y + 6(z+2)$$

$$(x-1) - y + 6(z+2) = x - y + 6z + 11$$

$$\boxed{x - y + 6z + 11 = 0}$$