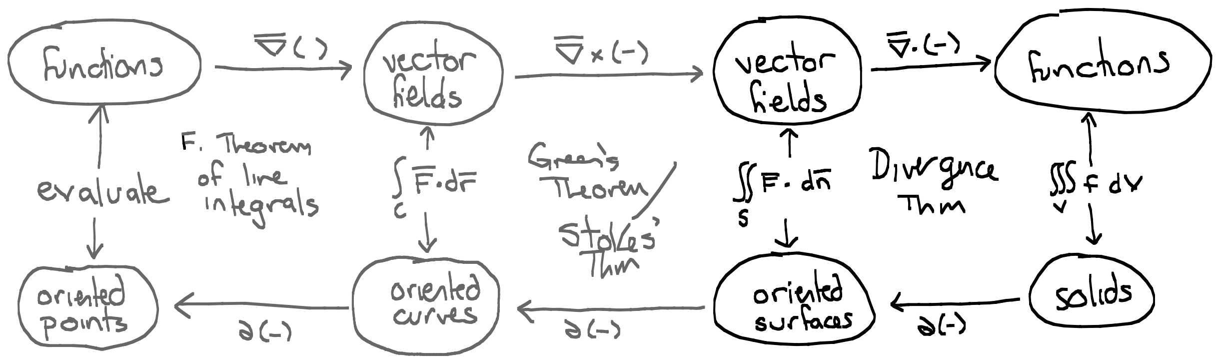
Lecture 27 - Divergence Theorem

Note Title



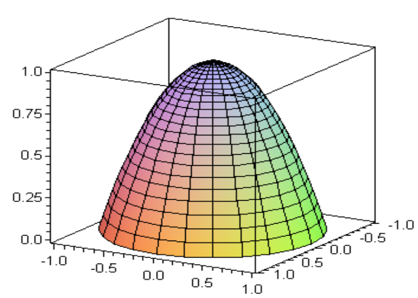
Thm If S is the boundary of a solid region R, endowed with the outward pointing normal and if $\overline{F}$ and all partials are defined on the interior of R, then

$$\iint_S \overline{F}\cdot d\overline{S} = \iiint_R \overline{\nabla}\cdot\overline{F}\, dV$$

In general, triple integrals are <u>much</u> easier to do than surface integrals.

Here the "outward pointing normal" is the one pointing out of the solid region (making it look like a hedgehog)

Ex Let R be the region $x^2+y^2\le 1$, $0\le z\le 1-x^2-y^2$, and let $S=\partial R$. Calculate $\iint_S \langle x,y,z\rangle\cdot d\overline{S}$ in two ways:



① Directly: S has two parts: $\underset{S_1}{z=1-x^2-y^2}$ and $\underset{S_2}{z=0}$.

Find $\iint_{S_1}$ and $\iint_{S_2}$ and add them.

S_1: Param by x & y: $\overline{n}=\langle 2x, 2y, 1\rangle$ $\left(=\overline{\nabla}(z-(1-x^2-y^2))\right)$

and $\overline{F}=\langle x,y,z\rangle=\langle x,y,1-x^2-y^2\rangle$. So

$$\iint_{S_1} \bar{F}\cdot d\bar{S} = \iint_{x^2+y^2\le 1} 2x^2+2y^2+(1-x^2-y^2)\, dA \underset{\text{go to polar}}{=} \int_0^{2\pi}\int_0^1 (1+r^2)\, r\, dr\, d\theta$$

$$= 2\pi\left(\frac{r^2}{2}+\frac{r^4}{4}\Big|_0^1\right) = \frac{3\pi}{2}.$$

S_2: In (x,y) plane, so $n = -\bar{k}$ (so it points outward)

Then $\bar{F} = \langle x,y,z\rangle = \langle x,y,0\rangle$; $\bar{F}\cdot\bar{n} = 0$

So $\iint_{S_2} \bar{F}\cdot d\bar{S} = 0 \Rightarrow \boxed{\iint_S \bar{F}\cdot d\bar{S} = \frac{3\pi}{2}}$

② Divergence Theorem:

$$\iint_S \bar{F}\cdot d\bar{S} = \iiint_R \bar{\nabla}\cdot\bar{F}\, dV$$

$\left\{\begin{matrix} x^2+y^2\le 1 \\ 0\le z\le 1-x^2-y^2\end{matrix}\right\} \leftrightarrow \left\{\begin{matrix} 0\le\theta<2\pi \\ 0\le r\le 1 \\ 0\le z\le 1-r^2\end{matrix}\right\}$ ← cylindrical works nicely

$$\bar{\nabla}\cdot\bar{F} = \frac{\partial}{\partial x}(x)+\frac{\partial}{\partial y}(y)+\frac{\partial}{\partial z}(z) = 3$$

$$\iiint_R \bar{\nabla}\cdot\bar{F}\, dV = \int_0^{2\pi}\int_0^1\int_0^{1-r^2} 3\, r\, dz\, dr\, d\theta$$

← much easier!

$$= 2\pi\int_0^1 3r-3r^3\, dr$$

$$= 3\pi-\frac{3}{2}\pi = \boxed{\frac{3\pi}{2}}$$

__Ex__: $R = [-1,1]\times[-1,1]\times[-1,1]$, $S = \partial R$, $\bar{F} = \langle x^2+\cos(yz), \ln(x^2+z^2+1)+y, e^{x^2y^2}\rangle$

Find $\iint_S \bar{F}\cdot d\bar{S}$.

Can do directly: S has 6 sides, so 6 integrals. ☹

Divergence Theorem:

$$\bar{\nabla}\cdot\bar{F} = \frac{\partial}{\partial x}\left(x^2+\cos(yz)\right)+\frac{\partial}{\partial y}\left(\ln(x^2+z^2+1)+y\right)+\frac{\partial}{\partial z}\left(e^{x^2y^2}\right) = 2x+1$$

$$\iint_S \bar{F}\cdot d\bar{S} = \int_{-1}^1\int_{-1}^1\int_{-1}^1 2x+1\, dx\, dy\, dz$$

$$= \boxed{8}.$$

← Much faster!

//

Ex: Let $\bar{x} = \langle x, y, z \rangle$. Let $\bar{F} = \frac{F \cdot \bar{x}}{|x|^3}$. Very important in physics! This gives inverse square fields.

① Gravity

② Electric

$$\frac{\partial}{\partial x}\left(\frac{F \cdot x}{(x^2+y^2+z^2)^{3/2}}\right) = F(x^2+y^2+z^2)^{-3/2} - 3Fx^2(x^2+y^2+z^2)^{-5/2}$$

$$= \frac{F(y^2+z^2-2x^2)}{(x^2+y^2+z^2)^{5/2}}$$

& similar for $\frac{\partial}{\partial y}$ & $\frac{\partial}{\partial z}$

So $\bar{\nabla} \cdot \bar{F} = 0$! The divergence theorem does not apply if the region contains $(0,0,0)$:

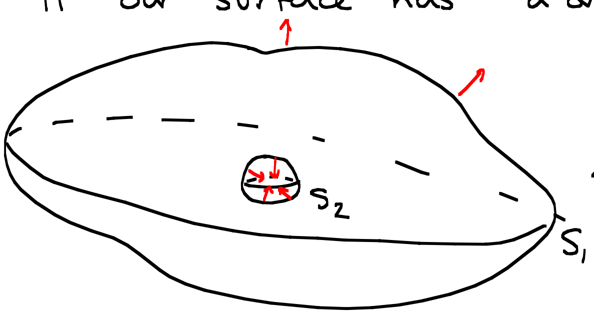
Let S = unit sphere. Then on S, $\bar{F} = F\langle x, y, z \rangle$ and $\bar{n} = \langle x, y, z \rangle$

If we use spherical coords, we get $\bar{F} \cdot d\bar{S} = F \sin\phi \, d\phi \, d\theta$.

$$\iint_S \bar{F} \cdot d\bar{S} = \int_0^{2\pi}\int_0^{\pi} F \sin\phi \, d\phi \, d\theta = 4\pi F.$$

So the divergence theorem doesn't apply.

If our surface has 2 or more components, can still use div. thm.



R the "cytoplasm" in this cell

S_1 oriented away from origin
S_2 oriented towards origin
} away from R

Then $$\iiint_R \bar{\nabla} \cdot \bar{F} \, dV = \iint_{S_1} \bar{F} \cdot d\bar{S} - \iint_{S_2} \bar{F} \cdot d\bar{S}$$

normal points in

So we learn that any surface containing the origin has
$$\iint_S \bar{F} \cdot d\bar{S} = 4\pi F$$ for the above example. This is Gauss' law!