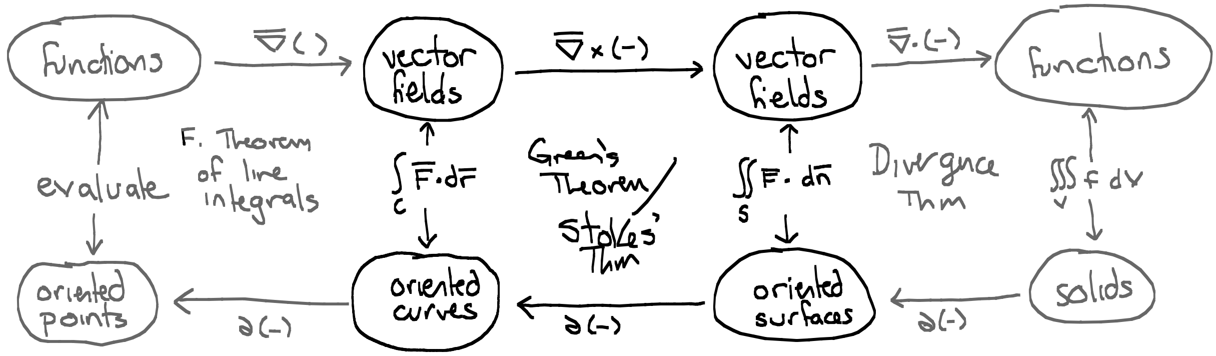
Lecture 25 - Stokes' Theorem

Note Title



<u>Thm</u> (Stokes' Theorem) Let S be an oriented surface with boundary curve C. Then

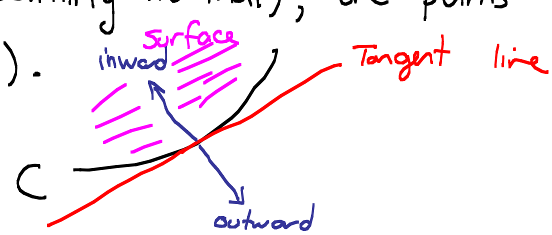
$$\iint_S \bar\nabla\times\bar F\cdot d\bar S = \int_C \bar F\cdot d\bar r$$

This has the same form as the F.T. of Line Ints: $\int_C \bar\nabla(f)\cdot dr =$ "eval f on ∂C & add"

Have to know how to orient C (otherwise the sign is wrong!)

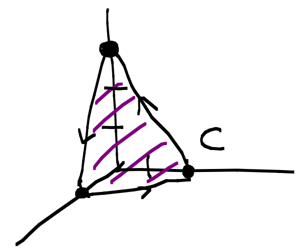
If S is an oriented surface, then $C = \partial S$ inherits an orientation by requiring that the surface be on the left if we walk along the curve with our head in the direction of the normal vector $\bar N$.

Make this precise by adding in the "outward pointing normal. Pick a point p on C and look at the tangent plane at p. In this sits the line tangent to C and 2 unit vectors $\perp$ $\bar T$. One points directly away from the surface (outward pointing normal), one points into the surface (inward pointing normal).



Then C is oriented by taking $\bar{T}$ in the direction $\bar{N} \times$ (outward pt. norm)
If S is the graph of a surface with the upward pointing normal, then C is oriented counterclockwise when we look from above.

Ex: C is the curve connecting $(1,0,0)$, $(0,2,0)$, and $(0,0,3)$, oriented as x-axis $\rightsquigarrow$ y-axis $\rightsquigarrow$ z-axis.



Curve is on the plane $6x+3y+2z=6$

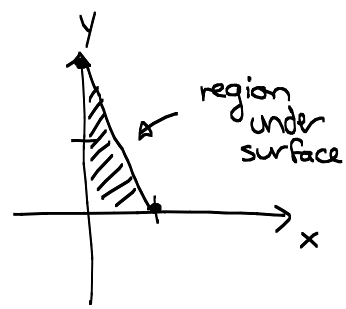
or $\langle \underbrace{3, \tfrac{3}{2}, 1}_{\bar{n}} \rangle \cdot \langle x,y,z \rangle = 3.$

Let $\bar{F}$ be $\langle e^{x^2}-y,\ 3z,\ 2x+\sin(\cos z) \rangle$

Stokes' Theorem makes $\int_C \bar{F} \cdot d\bar{r}$ a breeze:

C bounds the part of the plane in octant I.

$$\bar{\nabla} \times \bar{F} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \partial_x & \partial_y & \partial_z \\ e^{x^2}-y & 3z & 2x+\sin(\cos z) \end{vmatrix} = \langle -3, -2, 1 \rangle$$



Surface: $6x+3y+2z=6 \rightsquigarrow z = 6-3x-\frac{3}{2}y$

$\Rightarrow \bar{n} = \langle -\,{-3}, -\,{-\tfrac{3}{2}}, 1 \rangle$ ← vector from above. for graph of a surface, z-comp always ± 1!

$$\iint_S \bar{\nabla} \times \bar{F} \cdot d\bar{S} = \int_0^1 \int_0^{2-2x} \underbrace{(-9-3+1)}_{-11}\, dy\, dx$$

$$= \int_0^1 -22+22x \, dx = -22x+11x^2 \Big|_0^1 = \boxed{-11}.$$

Also makes surface integral easier:

Ex: $\bar{F} = \langle xe^{z}, y\cos z, zx^2y^3 \rangle$ S = upper unit hemisphere, upward normal

Find: $\iint_S \bar{\nabla} \times \bar{F} \cdot d\bar{S}$.

$$\bar{\nabla} \times \bar{F} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \partial_x & \partial_y & \partial_z \\ xe^{z} & y\cos z & zx^2y^3 \end{vmatrix} =$$

$$\langle 3zx^2y^2 + y\sin z,\ xe^{z} - 2zxy^3,\ 0 \rangle$$

S is a part of a sphere, so must use spherical coords.

$\Rightarrow$ messy!

Stokes': $\partial S =$ $z=0$ and $x^2+y^2=1$.

Oriented so that unit circle is traversed counterclockwise.

$$\left.\begin{array}{l} x=\cos t \\ y=\sin t \\ z=0 \end{array}\right\} \Rightarrow \bar{r}'(t) = \langle -\sin t, \cos t, 0\rangle$$

$$\bar{F} = \langle x, y, 0\rangle = \langle \cos t, \sin t, 0\rangle \Rightarrow \bar{F}\cdot\bar{r}'(t) = -\cos t \sin t + \cos t \sin t + 0$$

$$= 0.$$

$$\Rightarrow 0 = \int_C \bar{F}\cdot d\bar{r} = \iint_S \bar{\nabla}\times\bar{F}\cdot d\bar{S}$$

<u>Last Point</u>: Stokes' thm says that $\iint_S \bar{\nabla}\times\bar{F}\cdot d\bar{S}$ is independent of the surface, caring only about ∂S.

2 consequences:

① If S is a closed surface, then $\iint_S \bar{\nabla}\times\bar{F}\cdot d\bar{S} = 0$

② If we can find S_2 s.t. $\partial S_2 = \partial S$ &

$\iint_{S_2} \bar{\nabla}\times\bar{F}\cdot d\bar{S}$ is easier, then also get

$$\iint_S \bar{\nabla}\times\bar{F}\cdot d\bar{S} = \iint_{S_2} \bar{\nabla}\times\bar{F}\cdot d\bar{S}.$$

<u>Ex</u>: $S: \underset{x^2+y^2\le 1}{z = 1-x^2-y^2}$ $\bar{G} = \langle -x, y, 2\rangle \ (= \bar{\nabla}\times\langle -y, x, -xy\rangle)$

Then ① $\bar{n} = \langle 2x, 2y, 1\rangle$ (S is a surface)

$\Rightarrow \bar{G}\cdot\bar{n} = -2x^2 + 2y^2 + 2$ & can integrate.

② $\partial S =$ unit circle $= \partial$ unit disk. $= \partial S_2$

on S_2: $\bar{n} = \langle 0, 0, 1\rangle$, so $\bar{G}\cdot\bar{n} = 2$ &

$\iint_{S_2} \bar{G}\cdot\bar{n}\, dS = 2\cdot \text{area(disk)} = \boxed{2\pi}$. Easy!