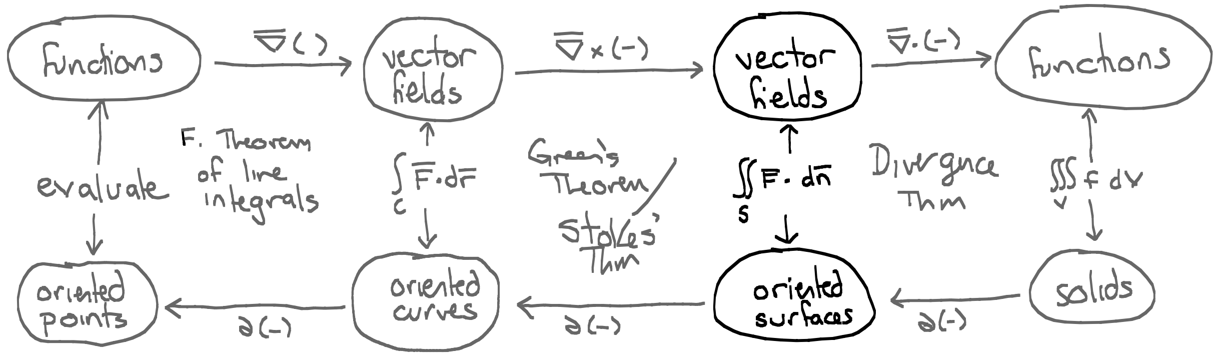
Note Title

Lecture 24 – Surface Integrals



Want to copy the notion of path integrals to surfaces:

1-D	2-D
curves	surfaces
$\bar{r}(t) \quad a\le t\le b$	$\bar{r}(u,v) \quad (u,v)$ in D
ds = element of arc length	dS = element of surface area
$\overline{T}$ = unit tangent	$\overline{N}$ = unit normal
$\int_C \overline{F}\cdot d\bar{s}$	$\iint_S \overline{F}\cdot d\overline{S}$

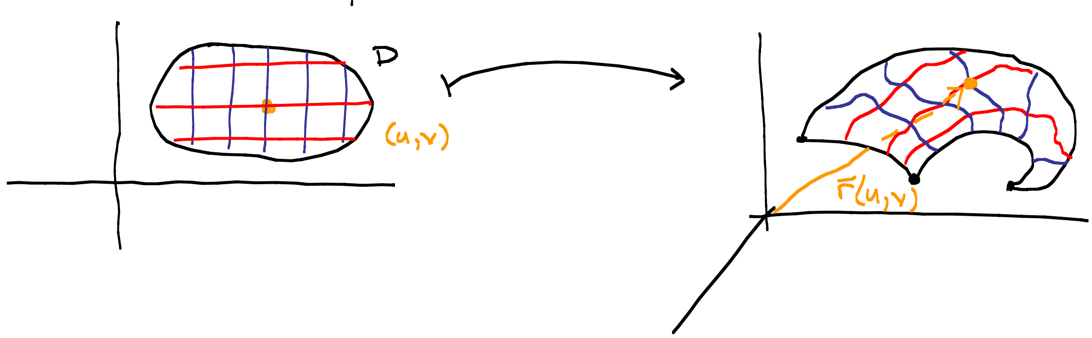
Parametric Surfaces:

Surfaces are 2-D, so we need 2 variables to describe them:

$\bar{r}(u,v) = \langle f(u,v), g(u,v), h(u,v)\rangle$ where (u,v) are in some region D in the (u,v)-plane

This is a parameterization: each point gets a (u,v) name.

Lines in (u,v)-plane become curves on our surface:



$\Rightarrow$ Get two families of curves:

$$\bar{r}(u,b) \quad \text{and} \quad \bar{r}(a,v) \quad \text{for fixed } (a,b) \text{ in } D.$$

Ex: Spherical coords parametrize the sphere: radius $= \rho = 1$

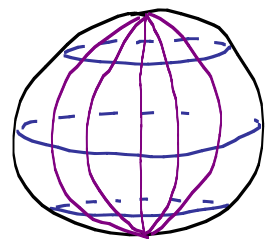
$$x = \sin\phi\cos\theta, \quad y = \sin\phi\sin\theta, \quad z = \cos\phi \qquad 0 \le \theta \le 2\pi, \quad 0 \le \phi \le \pi.$$

Curves $\theta = \alpha =$ fixed: •

lines of longitude

Curves $\phi = \beta =$ fixed: •

lines of latitude

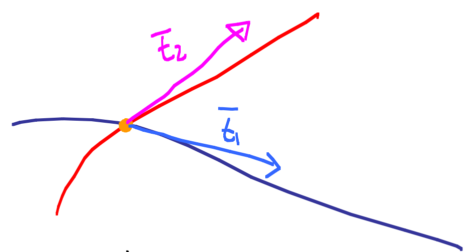


Fix a point (a,b) in D and consider $\bar{r}(a,v)$ and $\bar{r}(u,b)$.

We can find the tangent vectors at (a,b) to these curves:

$$\bar{t}_1 = \frac{d}{du}\bar{r}(u,b) = \frac{\partial \bar{r}}{\partial u}(a,b)$$

$$\bar{t}_2 = \frac{d}{dv}\bar{r}(a,v) = \frac{\partial \bar{r}}{\partial v}(a,b)$$



Ex: $\bar{r}(\theta,\phi) = \langle \cos\theta\sin\phi, \sin\theta\sin\phi, \cos\phi \rangle$

$$\frac{\partial r}{\partial \theta} : \langle -\sin\theta\sin\phi, \cos\theta\sin\phi, 0 \rangle \qquad \left(= \langle -y, x, 0 \rangle\right)$$

$$\frac{\partial r}{\partial \phi} : \langle \cos\theta\cos\phi, \sin\theta\cos\phi, -\sin\phi \rangle$$

This tells us the tangent plane at each point:

Def: $\bar{n} = \dfrac{\partial \bar{r}}{\partial u} \times \dfrac{\partial \bar{r}}{\partial v}$.

This is the normal vector to the tangent plane.

Def: $\bar{r}(u,v)$ is smooth if $|\bar{n}| \neq 0$.

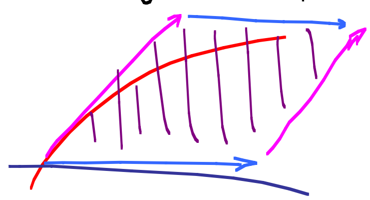
$\longleftrightarrow$ can define $\bar{N} = \dfrac{\bar{n}}{|\bar{n}|}$.

Def A surface is orientable if we can continuously choose $\bar{N}$.

If S = graph of $f(x,y)$: $\bar{r}(x,y) = \langle x, y, f(x,y) \rangle$

$\Rightarrow \bar{n} = \langle 1, 0, \frac{\partial f}{\partial x} \rangle \times \langle 0, 1, \frac{\partial f}{\partial y} \rangle = \langle -\frac{\partial f}{\partial x}, -\frac{\partial f}{\partial y}, 1 \rangle$

<u>Surface Integrals</u>: Look at $|\bar{n}|$: this is the area of the llogram spanned by $\frac{\partial \bar{r}}{\partial u}(a,b)$ and $\frac{\partial \bar{r}}{\partial v}(a,b)$



If we look at a tiny change in u & a tiny change in v, we get a small region on the surface.

If Δu & Δv are small enough, then this region looks like the llogram with sides $\frac{\partial \bar{r}}{\partial u} du$ & $\frac{\partial \bar{r}}{\partial v} dv$.

<u>Def</u> The <u>element of surface area</u> dS is

$$dS = \left| \frac{\partial \bar{r}}{\partial u} \times \frac{\partial \bar{r}}{\partial v} \right| du\, dv.$$

We form surface integrals just as we did path integrals: partition the surface, pick test points, & take a Riemann sum of $\sum f(x_i^*, y_i^*, z_i^*) \Delta S_i$.

If we have a parameterization: $\bar{r}(u,v)$, $(u,v) \in D$

$$\iint_S f\, dS = \iint_D f \cdot \left| \frac{\partial \bar{r}}{\partial u} \times \frac{\partial \bar{r}}{\partial v} \right| du\, dv$$

Can also do for vector fields: $d\bar{S} = \bar{N}\, dS$

So
$$\iint_S \bar{F} \cdot d\bar{S} = \iint_S \bar{F} \cdot \bar{N}\, dS$$

Since $\bar{N} = \frac{\bar{n}}{|\bar{n}|}$ and $dS = |\bar{n}|\, du\, dv$,

$$\iint_S \bar{F} \cdot d\bar{S} = \iint_D \bar{F} \cdot \left(\frac{\partial \bar{r}}{\partial u} \times \frac{\partial \bar{r}}{\partial v} \right) du\, dv$$