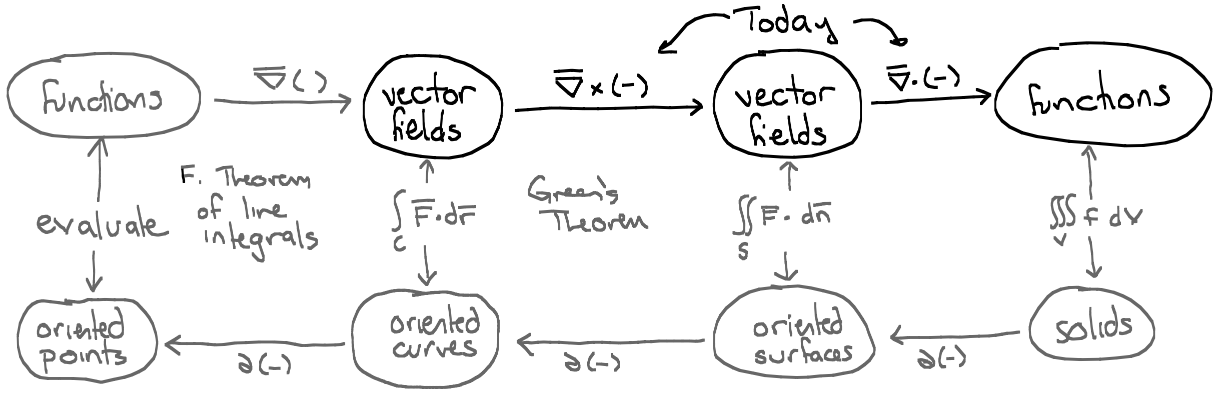
Lecture 23 - Curl & Divergence

Note Title

Common Picture:



Green's theorem fits into a bigger framework & we'll spend a few classes covering this.

Today: "How can we remember $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$?"

Two operations curl & divergence

Let $\bar{\nabla}$ denote the "vector" $\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \rangle$. Then "scalar multiplication by functions" $\longleftrightarrow$ gradient. Can do all other vector products.

Def The curl of a vector field $\bar{F} = P\bar{i} + Q\bar{j} + R\bar{k}$ is

$$\bar{\nabla}\times\bar{F} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\bar{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\bar{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\bar{k}$$

The final formula is very hard to remember, so it is easier to remember this as the cross product $\bar{\nabla}\times\bar{F}$.

Must remember: stuff from $\bar{\nabla}$ always acts on stuff from $\bar{F}$. So

$\frac{\partial Q}{\partial z}$, not $Q\frac{\partial}{\partial z}$!

Ex: $\bar{F} = \langle -y, x, xy \rangle \Rightarrow \operatorname{curl}(\bar{F}) = \bar{\nabla}\times\bar{F} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y & x & xy \end{vmatrix}$

$$= \left(\frac{\partial}{\partial y}(xy) - \frac{\partial}{\partial z}x\right)\bar{i} + \left(\frac{\partial}{\partial z}(-y) - \frac{\partial}{\partial x}(xy)\right)\bar{j} + \left(\frac{\partial}{\partial x}(x) - \frac{\partial}{\partial y}(-y)\right)\bar{k} = \langle x, -y, 2 \rangle$$

Prop $\bar\nabla \times (\bar\nabla f) = \bar 0$.

Just check $\bar k$ component of $\begin{vmatrix} \bar i & \bar j & \bar k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{vmatrix}$:
(others are similar)

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) - \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x \partial y} - \frac{\partial^2 f}{\partial y \partial x} = 0$$ by Clairault's Theorem.

Can generalize an earlier theorem:

On a simply connected region,

$\bar F$ is conservative $\longleftrightarrow$ $\bar\nabla \times \bar F = \bar 0$.

As the name implies, curl measures rotation. We should think of the direction of $\bar\nabla \times \bar F$ as the axis of rotation & $|\bar\nabla \times \bar F|$ as the speed.

<u>Ex</u>: $\bar F = \langle -y, x, 0 \rangle$ then $\bar\nabla \times \bar F = \langle 0, 0, 2 \rangle$, and $\bar F$ visibly rotates counterclockwise about the z-axis.

<u>Def</u> If $\bar\nabla \times \bar F = \bar 0$, say $\bar F$ is <u>irrotational</u>.

So $\bar\nabla f$ is always irrotational.

<u>Def</u> The <u>divergence</u> of $\bar F$ is

$$\bar\nabla \cdot \bar F = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}.$$

<u>Ex</u> $\bar F = \langle x, y, z \rangle \Rightarrow \bar\nabla \cdot \bar F = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z) = 3$

$\bar F = \langle -y, x, z^2 \rangle \Rightarrow \bar\nabla \cdot \bar F = \frac{\partial}{\partial x}(-y) + \frac{\partial}{\partial y}(x) + \frac{\partial}{\partial z}(z^2) = 2z$

Just as with curl & grad, have a rel between div & curl:

<u>Prop</u>: $\bar\nabla \cdot (\bar\nabla \times \bar F) = 0$

We would expect this: $\bar\nabla \times \bar F$ should be $\perp$ to $\bar\nabla$.

<u>Pf</u>: $\bar\nabla \times \bar F = \left\langle \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}, \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}, \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right\rangle$

$$\bar\nabla \cdot (\bar\nabla \times \bar F) = \frac{\partial}{\partial x}\left(\cancel{\frac{\partial R}{\partial y}} - \cancel{\frac{\partial Q}{\partial z}}\right) + \frac{\partial}{\partial y}\left(\cancel{\frac{\partial P}{\partial z}} - \cancel{\frac{\partial R}{\partial x}}\right) + \frac{\partial}{\partial z}\left(\cancel{\frac{\partial Q}{\partial x}} - \cancel{\frac{\partial P}{\partial y}}\right) = 0$$

<u>Def</u> $\bar F$ is <u>incompressible</u> if $\bar\nabla \cdot \bar F = 0$

So $\bar{\nabla}\times\bar{F}$ is always incompressible.

Terminology comes from what $\bar{\nabla}\cdot(-)$ measures: radial expansion and contraction of the vector field.

<u>Ex</u>: $\bar{F}=\langle x,-y,1\rangle \Rightarrow \bar{\nabla}\cdot\bar{F}=\frac{\partial}{\partial x}(x)+\frac{\partial}{\partial y}(-y)+\frac{\partial}{\partial z}(1)=1-1=0.$

x dir expanding out at same rate y is shrinking in.

<u>Ex</u>: $\bar{F}=\langle x,y,z\rangle \Rightarrow \bar{\nabla}\cdot\bar{F}=3$ and $\bar{F}$ points out everywhere.

<u>Note</u> It is not always the case that $\bar{\nabla}\cdot\bar{F}=0 \Rightarrow \bar{F}=\bar{\nabla}\times\bar{G}$. This requires that we have no "2 d" holes.

This is conceptually parallel to the bottom row of our chart:

The boundary of a surface is a closed curve (no boundary)

The boundary of a solid region is a closed surface (no boundary)

Have one final combination:

<u>Def</u> The <u>Laplacian</u> of f is

$$\nabla^2 f=\bar{\nabla}\cdot\bar{\nabla}f=\frac{\partial^2 f}{\partial x\partial x}+\frac{\partial^2 f}{\partial y\partial y}+\frac{\partial^2 f}{\partial z\partial z}.$$

The Laplacian is not always zero!

<u>Ex</u> $f=x^2+y^2+z^2 \Rightarrow \nabla^2 f=6.$

This is very important in physics: ∇^2 in some sense measures "diffusion". If $\nabla^2 f=0$, say f is <u>harmonic</u>.

<u>Ex</u>: $f(x,y,z)=x+2y+3z$ is harmonic.

<u>Longer example</u>:

$\bar{F}=\langle x,z,y\rangle$ then $\bar{\nabla}\times\bar{F}=\langle 0,0,0\rangle$ so should be able to find

f s.t. $\bar{\nabla}f=\bar{F}$. $\frac{\partial f}{\partial x}=x \Rightarrow f=\frac{1}{2}x^2+g(x,y)$

$\frac{\partial f}{\partial y}=z=\frac{\partial g}{\partial y} \Rightarrow g(y,z)=yz+h(z)$. $\frac{\partial f}{\partial z}=y=y+h'(z)\Rightarrow h'(z)=0.$

So $f(x,y,z)=\frac{1}{2}x^2+yz.$

$\bar{\nabla}\cdot\bar{F}=1.$