Lecture 21 – Fundamental Theorem of Line Integrals

Note Title

Remaining Lectures have a common theme. Best summerized with a picture we'll fill in:

$$\begin{tikzcd}[column sep=large, row sep=large]
\text{functions} \arrow[r, "\nabla(-)"] \arrow[d, leftrightarrow, "\text{evaluate}"'] \arrow[dr, phantom, "\text{F. Theorem of line integrals}"] & \text{vector fields} \arrow[r, "\nabla\times(-)"] \arrow[d, leftrightarrow, "\int_C \bar{F}\cdot d\bar{r}"] & \text{vector fields} \arrow[r, "\nabla\cdot(-)"] \arrow[d, leftrightarrow, "\iint_S \bar{F}\cdot d\bar{n}"] & \text{functions} \arrow[d, leftrightarrow, "\iiint_V f\, dV"] \\
\text{oriented points} & \text{oriented curves} \arrow[l, "\partial(-)"] & \text{oriented surfaces} \arrow[l, "\partial(-)"] & \text{solids} \arrow[l, "\partial(-)"]
\end{tikzcd}$$

First: the Fundamental theorem of line integrals.

<u>Thm</u> Let C be a path from (a,b,c) to (x,y,z) then

$$\int_C \bar{\nabla} f \cdot d\bar{r} = f(x,y,z) - f(a,b,c)$$

<u>Pf</u>: $\bar{\nabla} f \cdot \bar{r}'(t) = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle \cdot \left\langle \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \right\rangle$

$$= \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}$$

(chain rule) $= \frac{d}{dt}(f)$.

F. Thm of Calc.

So $\displaystyle\int_C \bar{\nabla} f \cdot d\bar{r} = \int_\alpha^\beta \frac{d}{dt} f \, dt = f\big(x(\beta), y(\beta), z(\beta)\big) - f\big(x(\alpha), y(\alpha), z(\alpha)\big)$

$$= f(x,y,z) - f(a,b,c).$$

<u>Ex</u>: $f(x,y,z) = e^{x^2} + y/z$

C the curve $\bar{r}(t) = \langle e^t, \sin t, \cos t \rangle \quad 0 \le t \le \pi/4$

Then $\displaystyle\int_C \bar{\nabla} f \cdot d\bar{r} = f(\bar{r}(\pi/4)) - f(\bar{r}(0)) = \left(e^{\pi^2/16} + 1\right) - (1) = e^{\pi^2/16}.$

In particular, we see that

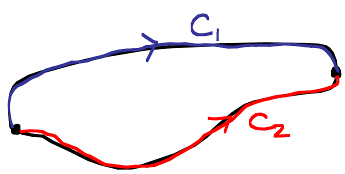
$\int_C \bar{\nabla} f \cdot d\bar{r}$ does not depend on the path taken between to endpoints.

Def: $\int_C \bar{F}\cdot d\bar{r}$ is path independent if it depends only on the endpoints of C.

So $\bar{\nabla} f$ gives path independent integrals.

Thm $\int \bar{F}\cdot d\bar{r}$ is path independent if $\int_C \bar{F}\cdot d\bar{r} = 0$ for every simple closed curve (a curve that hits itself only at the endpoints).

Why? A simple closed curve is 2 paths:



$C = C_1 \cup -C_2$

So if $\int_C \bar{F}\cdot d\bar{r}$ is zero, then

$$\int_{C_1} \bar{F}\cdot d\bar{r} - \int_{C_2} \bar{F}\cdot d\bar{r} = 0.$$

So we learn that

$$\bar{F} = \bar{\nabla} f \implies \int_C \bar{F}\cdot d\bar{r} \text{ is path ind.} \iff \int_C \bar{F}\cdot d\bar{r} = 0 \text{ for closed } C$$

Ex: Saw last time that $\bar{F} = \langle -y, x\rangle$ has

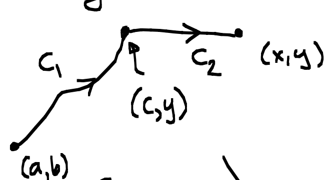
$\int_C \bar{F}\cdot d\bar{r} = 2\pi$ when $C =$ unit circle.

$\Rightarrow$ $\bar{F}$ is not conservative.

Can do slightly better: $\bar{F} = \bar{\nabla} f \longleftrightarrow \int_C \bar{F}\cdot d\bar{r}$ is path ind.

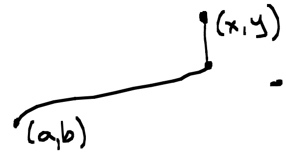
Idea: Pick a base point (a,b). Then $\int_C \bar{F}\cdot d\bar{r}$ depends only on the end point (x,y): Call it $f(x,y)$.

Take two paths:



Then $$\frac{\partial f}{\partial x}(x,y) = \frac{\partial}{\partial x}\left(\int_{C_1} \bar{F}\cdot d\bar{r} + \int_{C_2} \bar{F}\cdot d\bar{r}\right) = \frac{\partial}{\partial x}\left(\int_{C_2} P\,dx + Q\,dy\right)$$

On C_2: $x = x$, y fixed $\Rightarrow$ $dx = dx$, $dy = 0$ $\Rightarrow$ $= \frac{\partial}{\partial x}\int_c^x P\,dx = P.$

For $\frac{\partial}{\partial y}$: take 

So conservative $\longleftrightarrow$ path independent.

Still hard to check. If $\bar{F} = \langle P, Q \rangle = \bar{\nabla} f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle$

then $$\frac{\partial P}{\partial y} = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial Q}{\partial x}.$$

So see easily that $\langle -y, x \rangle$ is not conservative.

Converse is harder: need conditions on our region.

<u>Def</u> A subset of $\mathbb{R}^2$ is <u>simply connected</u> if it is connected and every loop in the subset encloses only points in the subset.

In other words, no holes:  not conn.  not simply connected

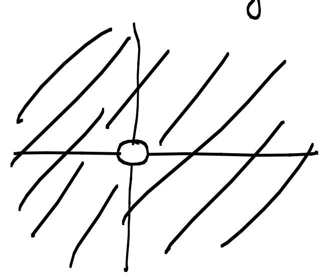
<u>Thm</u> If D is a simply connected region, then if $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ on D, then $\langle P, Q \rangle = \bar{\nabla} f$.

Simple Connectivity is essential!

$$\bar{F} = \left\langle \frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \right\rangle \quad \text{then} \quad \frac{\partial Q}{\partial x} = \frac{(x^2+y^2) - 2x^2}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$$

$$\frac{\partial P}{\partial y} = \frac{-(x^2+y^2)+2y^2}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2} = \frac{\partial Q}{\partial x}$$

<u>But!</u> $\int_C \bar{F} \cdot d\bar{r} = 2\pi$ if C is the unit circle. Why? $\bar{F}$ is only defined on $\mathbb{R}^2 - (0,0)$:



This is <u>not</u> simply connected, so previous thm doesn't apply. In fact, this detects the failure of simple connectivity!