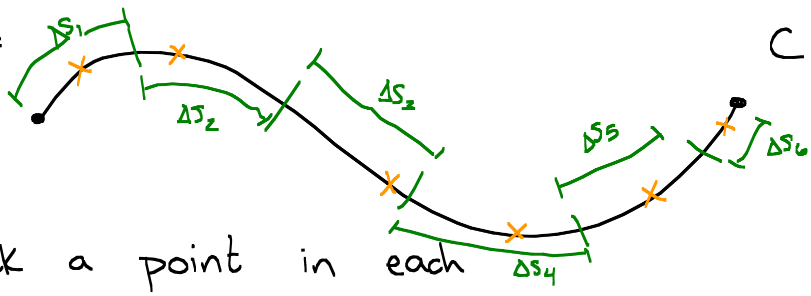
Lecture 20 – Line Integrals

Note Title

Start generalizing basic notions of integrals to work over other regions.

Given a curve C in space:



and a function $f(x,y,z)$, can integrate f over C:

$\int_C f\, ds$. Subdivide C & pick a point in each subdivision.

Δs_i = length of i^{th} subdivision

(x_i^*, y_i^*, z_i^*) = test pt therein

Then $$\int_C f\, ds = \lim \sum f(x_i^*, y_i^*, z_i^*)\,\Delta s_i$$

where the limit is as $\Delta s \to 0$. This is a line integral.

Geometric story: Imagine a curtain whose lower hem is along the curve and whose height at each point is $f(x,y,z)$. Then $\int_C f\, ds$ is the area of the curtain.

Never compute with Riemann sum! Parametrize C: $\vec{r}(t) = \langle x(t), y(t), z(t)\rangle$

$a \le t \le b$

$$\Rightarrow \quad ds = \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}\, dt$$

$f(x,y,z) = f(x(t), y(t), z(t))$ so

$$\int_C f(x,y,z)\, ds = \int_a^b f(x(t), y(t), z(t)) \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}\, dt$$

Ex: $f(x,y,z) = z$ C the helix given by

$\vec{r}(t) = \langle \cos t, \sin t, t\rangle$ $\Rightarrow$ $ds = |\vec{r}\,'(t)| = \sqrt{(-\sin t)^2 + (\cos t)^2 + 1}\, dt = \sqrt{2}\, dt$

$0 \le t \le 2\pi$

$$\int_C f\,ds = \int_0^{2\pi} t\cdot\sqrt{2}\,dt = \frac{\sqrt{2}}{2}t^2\Big]_0^{2\pi} = 2\sqrt{2}\,\pi^2$$

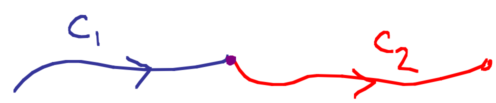
Can also form $\int_C f\,dx$, $\int_C f\,dy$, $\int_C f\,dz$.

All are computed the same way.

These depend on the orientation of C. $\int_C f\,ds$ does not

Properties:

① If $C = C_1 \cup C_2$:



Then $\int_C f\,ds = \int_{C_1} f\,ds + \int_{C_2} f\,ds$ etc.

② If we reverse the orientation: $C \to -C$,

we ⓐ leave $\int ds$ unchanged: $\int_C f\,ds = \int_{-C} f\,ds$

ⓑ Swap sign of $\int dx, \ldots$: $\int_C f\,dx = -\int_C f\,dx$

There is a second kind of line integral: integrals of vector fields.

Def: $d\bar{r} = \langle dx, dy, dz\rangle$ if $\bar{F} = \langle P, Q, R\rangle$, then can define

$$\int_C \bar{F}\cdot d\bar{r} = \int_C P\,dx + Q\,dy + R\,dz.$$

With a parametrization of C: $\bar{r}(t) = \langle f(t), g(t), h(t)\rangle$ $\quad a \le t \le b$

Then $d\bar{r} = \langle dx, dy, dz\rangle = \langle f'(t), g'(t), h'(t)\rangle\,dt$

Have another, intrinsic form of line integrals:

$$d\bar{r} = \bar{T}\,ds$$

where $\bar{T}$ is the unit tangent vector to the curve.

Why? $\bar{T} = \dfrac{\bar{r}'(t)}{|\bar{r}'(t)|}$, while $ds = |\bar{r}'(t)|\,dt$ so

$$\bar{T}\,ds = \frac{\bar{r}'(t)}{\cancel{|\bar{r}'(t)|}}\cdot \cancel{|\bar{r}'(t)|}\,dt = \bar{r}'(t)\,dt = d\bar{r}.$$

This explains some of the sign change: Changing orientation swaps

$\overline{T}$ with $-\overline{T}$ (an orientation is a choice of $\overline{T}$).

<u>Ex</u>: $\overline{F} = \langle -y, x \rangle$ C = unit circle, oriented counterclockwise.

$\bar{r}(t) = \langle \cos t, \sin t \rangle$ $0 \le t \le 2\pi$

$\overline{T}(t) = \langle -\sin t, \cos t \rangle = \bar{r}'(t)$ (and $ds = dt$)

Then $$\int_C \overline{F} \cdot d\bar{r} = \int_C -y\, dx + x\, dy = \int_0^{2\pi} (-\sin t)(-\sin t) + (\cos t)(\cos t)\, dt$$
$$= \boxed{2\pi}$$

<u>Ex</u>: $\overline{F} = \langle x, y \rangle$ C = unit circle, oriented counterclockwise

$$\int_C \overline{F} \cdot d\bar{r} = \int_C x\, dx + y\, dy = \int_0^{2\pi} (\cos t)(-\sin t) + (\sin t)(\cos t)\, dt = 0.$$

<u>Physically</u>: $\overline{F}$ is a "force field" like gravity, EM, etc.

C a path a particle takes through the field. Then

$\int_C \overline{F} \cdot d\bar{r}$ is the "work" done moving the particle.