Lecture 2 – Dot & Cross Products

Note Title

Today: More geometry!

Def If $\bar{u} = \langle a,b,c\rangle$ and $\bar{v} = \langle x,y,z\rangle$ then the <u>dot product</u> (= <u>scalar</u> <u>prod</u>) of $\bar{u}$ and $\bar{v}$ is

$$\bar{u}\cdot\bar{v} = ax + by + cz.$$

<u>Ex</u> $\bar{u} = \langle 1,1,1\rangle$, $\bar{v} = \langle -3,1,-2\rangle$, then

$$\bar{u}\cdot\bar{v} = 1\cdot(-3) + 1\cdot(1) + 1\cdot(-2) = -4$$

Has nice, usual properties: $\bar{u}\cdot\bar{v} = \bar{v}\cdot\bar{u}$, $\cdot$ distributes over $+$,

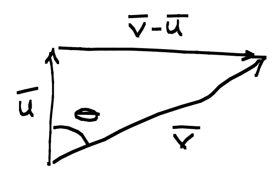
$$(a\bar{u})\cdot\bar{v} = \bar{u}\cdot(a\bar{v}) = a(\bar{u}\cdot\bar{v}).$$

This is <u>very</u> easy to compute and has 2 main ties to geometry.

1. If $\bar{u} = \langle a,b,c\rangle$, then $\bar{u}\cdot\bar{u} = a^2+b^2+c^2 = |\bar{u}|^2$!
2. If θ is the angle between $\bar{u}$ and $\bar{v}$, then

$$\cos\theta = \frac{\bar{u}\cdot\bar{v}}{|\bar{u}|\cdot|\bar{v}|}$$

2 is secretly the law of cosines:



$$(\bar{v}-\bar{u})\cdot(\bar{v}-\bar{u}) = |\bar{v}-\bar{u}|^2 = |\bar{v}|^2 + |\bar{u}|^2 - 2|\bar{u}||\bar{v}|\cos\theta$$

So $\cdot$ tells us about the underlying geometry.

<u>Ex</u>: $\bar{u} = \langle 1,1,1\rangle$, $\bar{v} = \langle -3,1,-2\rangle$

so $|\bar{u}| = \sqrt{3}$, $|\bar{v}| = \sqrt{14}$, and

$$\cos\theta = \frac{-2}{\sqrt{42}}$$

Can rewrite the formula for $\cos\theta$:

$$\cos\theta = \frac{\bar{u}\cdot\bar{v}}{|\bar{u}|\cdot|\bar{v}|} = \left(\frac{\bar{u}}{|\bar{u}|}\right)\cdot\left(\frac{\bar{v}}{|\bar{v}|}\right)$$

shorthand for $\frac{1}{|\bar{v}|}\cdot\bar{v}$

Vectors of the form $\frac{\bar{v}}{|\bar{v}|}$ are special: they have length 1.

Def A <u>unit</u> <u>vector</u> is one of length 1.

Unit vectors only carry directional information, and this is all we needed to get the angle between vectors.

Given any $\bar{v} \neq \bar{0}$, get a unit vector $\bar{u}_{\bar{v}} = \dfrac{\bar{v}}{|\bar{v}|}$.

Ex

$\bar{v} = \langle 0,3,4 \rangle$, so $|\bar{v}| = \sqrt{9+16} = 5$; $\bar{u}_{\bar{v}} = \langle 0, 3/5, 4/5 \rangle$

$\bar{w} = \langle 4,-3,0 \rangle$, then $|\bar{w}| = 5$; $\bar{u}_{\bar{w}} = \langle 4/5, -3/5, 0 \rangle$

$\bar{v} \cdot \bar{w} = 0 \cdot 4 + 3 \cdot (-3) + 4 \cdot 0 = -9$, so

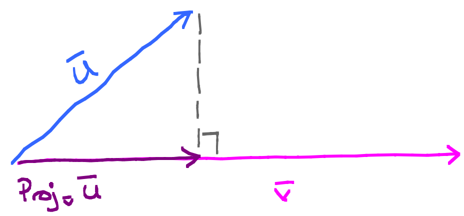
$$\cos\theta = \frac{-9}{5 \cdot 5} = 0 \cdot (4/5) + (3/5)(-3/5) + (4/5) \cdot 0 .$$

Def $\bar{u}$ and $\bar{v}$ are orthogonal if $\bar{u} \cdot \bar{v} = 0$. ($\bar{u} \cdot \bar{v} = 0 \Rightarrow \cos\theta = 0 \Rightarrow \theta = \pi/2$)

So $\langle 1,1,1 \rangle$ and $\langle -3,1,-2 \rangle$ are not orthogonal, but $\langle 1,1,1 \rangle$ and $\langle -3,1,2 \rangle$ are.

Dot product also gives projections:

$$\mathrm{Proj}_{\bar{v}}\, \bar{u} = \frac{\bar{u} \cdot \bar{v}}{\bar{v} \cdot \bar{v}} \cdot \bar{v}$$



$\bar{u} = \langle 1,1,1 \rangle$

$\bar{v} = \langle 1,1,-1 \rangle$

$$\left.\begin{array}{l} \bar{u} \cdot \bar{v} = 1 \\ \bar{v} \cdot \bar{v} = 3 \end{array}\right\} \Rightarrow \mathrm{Proj}_{\bar{v}}\, \bar{u} = \frac{1}{3}\bar{v} = \langle 1/3, 1/3, -1/3 \rangle$$

Have 3 very special unit vectors:

$\bar{i} = \langle 1,0,0 \rangle$ $\quad \bar{j} = \langle 0,1,0 \rangle$ $\quad \bar{k} = \langle 0,0,1 \rangle$

If $\bar{u} = \langle a,b,c \rangle$, then $\bar{u} = a\bar{i} + b\bar{j} + c\bar{k}$. ← Really useful!

<u>Cross Product</u> (= <u>vector product</u>.)

Two defs:

① Geometric given $\bar{u}$ & $\bar{v}$, $\bar{u} \times \bar{v}$ is the vector whose magnitude is $|\bar{u}| \cdot |\bar{v}| \cdot \sin\theta$, where $\cos\theta = \dfrac{\bar{u} \cdot \bar{v}}{|\bar{u}| \cdot |\bar{v}|}$, and which is orthogonal to $\bar{u}$ and $\bar{v}$.

This is actually ambiguous: there are 2 ⇒ Right-hand rule.

$\underline{\text{Ex}}$ $\quad \bar{u} = \langle 1,2,0 \rangle, \quad \bar{v} = \langle 2,1,0 \rangle \quad$ then $\quad |\bar{u}| = |\bar{v}| = \sqrt{5}, \quad \bar{u}\cdot\bar{v} = 4,$

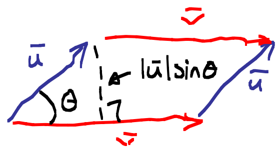
so $\quad \cos\theta = \frac{4}{5} \Rightarrow \sin\theta = 3/5 \quad$ & $\quad |\bar{u}\times\bar{v}| = 3$

direction : $\langle 0,0,-1 \rangle$, so

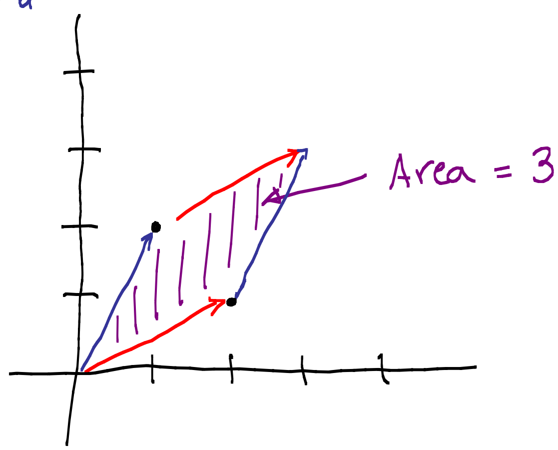
$$\bar{u}\times\bar{v} = \langle 0,0,-3 \rangle.$$

- If $\bar{u}$ & $\bar{v}$ are $\parallel$, then $\bar{u}\times\bar{v} = \bar{0}$ $\quad (\cos\theta = \pm 1 \Rightarrow \theta = 0 \text{ or } \pi \Rightarrow \sin\theta = 0)$
- $\bar{u}\times\bar{v} = -(\bar{v}\times\bar{u})$

- x - also has geometric content : $|\bar{u}\times\bar{v}|$ = area of the $\parallel$ogram with sides $\bar{u}$ & $\bar{v}$.



$\underline{\text{Ex}}$: $\langle 1,2,0 \rangle$
$\langle 2,1,0 \rangle$



$\underline{\text{Ex}}$: $\quad \bar{i}, \bar{j}, \bar{k}$: $\quad$ all orthogonal, all unit vectors

$\bar{i}\times\bar{j} = \bar{k}, \quad \bar{j}\times\bar{k} = \bar{i}, \quad \bar{k}\times\bar{i} = \bar{j}$: