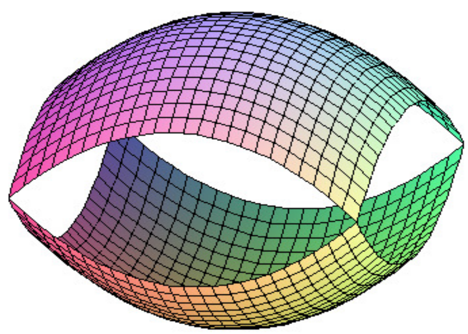
Lecture 18 - Triple Integrals & Spherical Coords

Note Title

<u>Ex</u>: Find the volume of the region bound between $z = 1-x^2-y^2$, $z = x^2+y^2-1$ and $-\sqrt{2}/2 \le \begin{matrix} x \\ y \end{matrix} \le \sqrt{2}/2$.

x & y bound by numbers: $\left.\begin{matrix} -\sqrt{2}/2 \le x \le \sqrt{2}/2 \\ -\sqrt{2}/2 \le y \le \sqrt{2}/2 \end{matrix}\right\}$ any order

z bound by functions $x^2+y^2-1 \le z \le 1-x^2-y^2$ } first



→ Volume: $$\int_{-\sqrt{2}/2}^{\sqrt{2}/2}\int_{-\sqrt{2}/2}^{\sqrt{2}/2}\int_{x^2+y^2-1}^{1-x^2-y^2} dz\, \underbrace{dx\, dy}_{\text{order doesn't matter}}$$

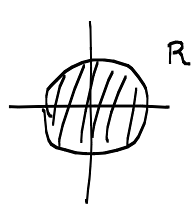
$$= \int_{-\sqrt{2}/2}^{\sqrt{2}/2}\int_{-\sqrt{2}/2}^{\sqrt{2}/2} 2-2x^2-2y^2\, dx\, dy = 8\int_0^{\sqrt{2}/2}\int_0^{\sqrt{2}/2} 1-x^2-y^2\, dx\, dy = \frac{8}{3}$$

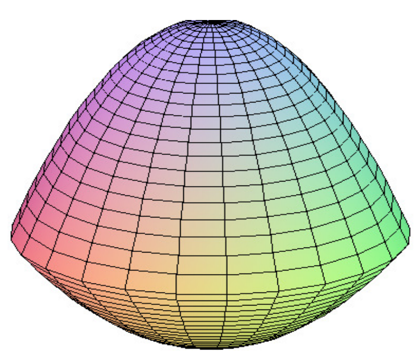
<u>Ex</u>: Set up an integral giving the volume between $z = 2-2x^2-2y^2$ and $z = x^2+y^2-1$.

Two surfaces hit at

$x^2+y^2-1 = 2-2x^2-2y^2 \iff$

$x^2+y^2 = 1$





Region: $x^2+y^2-1 \le z \le 2-2x^2-2y^2$

$x^2+y^2 \le 1 \leftarrow$ $-\sqrt{1-x^2} \le y \le \sqrt{1-x^2}$, $-1 \le x \le 1$

So the volume is given by

$$\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{x^2+y^2-1}^{2-2x^2-2y^2} dz\,dy\,dx$$

Have 6 kinds of regions: same idea w/ 2-var:

1st var bound by functions of other 2

2nd var bound by functions of 3rd

3rd var bound by numbers

Ex:

These first

$a(y,z) \le x \le b(y,z) \to$
- $e(y) \le z \le f(y)$, $c \le y \le d$
- $c(z) \le y \le d(z)$, $e \le z \le f$

$c(x,z) \le y \le d(x,z) \to$
- $e(x) \le z \le f(x)$, $a \le x \le b$
- $a(z) \le x \le b(z)$, $e \le z \le f$

$e(x,y) \le z \le f(x,y) \to$
- $a(y) \le x \le b(y)$, $c \le y \le d$
- $c(x) \le y \le d(x)$, $a \le x \le b$

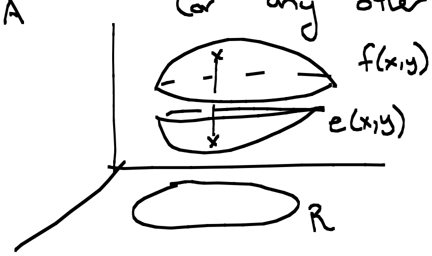
These second

Otherwise, the set-up is exactly the same as for the three variable version.

Can also interpret these as an iterated double & single integral:

$$\iiint_V g\,dV = \iint_R \left(\int_e^f g\,dz\right)dA$$

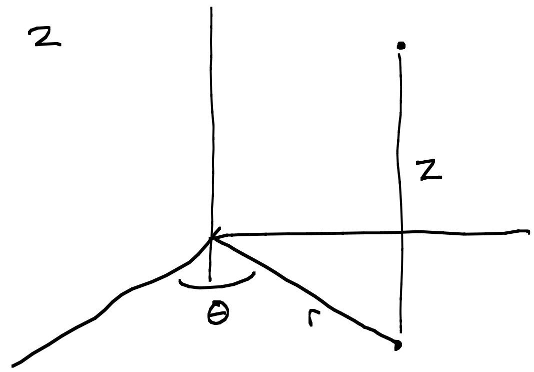
(or any other order)



We can use "cylindrical" coords for triple integrals:

= polar for (x,y) & ordinary z

$(r,\theta,z) \longleftrightarrow (x,y,z)$



In cylindrical: $dV = r\,dz\,dr\,d\theta$

Ex: Volume of a sphere:

$$-\sqrt{1-x^2-y^2} \le z \le \sqrt{1-x^2-y^2}$$
$$-\sqrt{1-r^2} \le z \le \sqrt{1-r^2}$$
$$0 \le r \le 1$$
$$0 \le \theta \le 2\pi$$

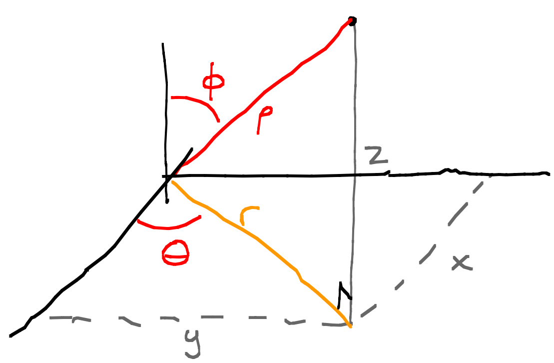
$$V = \int_0^{2\pi}\int_0^1\int_{-\sqrt{1-r^2}}^{\sqrt{1-r^2}} r\,dz\,dr\,d\theta$$

$$= 2\pi \int_0^1 2r\sqrt{1-r^2}\,dr$$

$u = 1-r^2$
$du = -2r\,dr$

$$= 2\pi\int_0^1 \sqrt{u}\,du = \frac{4}{3}\pi.$$

Polar has a second analogue: spherical



ρ is the "azmuthal angle" & from the picture, we get formulas:

$$z = \rho\cos\phi$$

$$r = \rho\sin\phi \nearrow \begin{matrix} x = \rho\sin\phi\cos\theta \\ y = \rho\sin\phi\sin\theta \end{matrix}$$