Lecture 17 – Change of Rings ; Triple Integrals

Note Title

Ex: ① $x = ar\cos\theta$, $y = br\sin\theta$ $(a,b>0)$ — elliptic coords

then $\dfrac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} a\cos\theta & -ar\sin\theta \\ b\sin\theta & br\cos\theta \end{vmatrix} = abr$

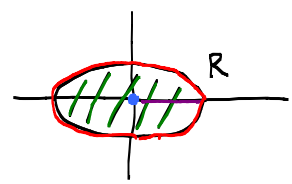
So $dA = abr\,dr\,d\theta$

② $x = u^2 - v^2$, $y = uv$ then $\dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 2u & -2v \\ v & u \end{vmatrix} = 2u^2 + 2v^2$

So $dA = 2u^2 + 2v^2\,du\,dv$

Need now to understand how R changes.

Ex R: $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} \le 1$ plug in $\left\{\begin{matrix} x = ar\cos\theta \\ y = br\sin\theta \end{matrix}\right\}$

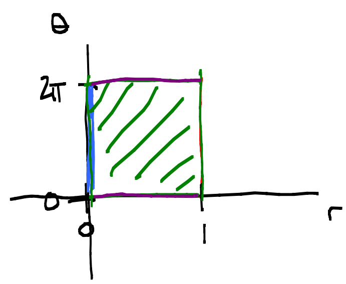


$\Rightarrow r^2 \le 1$

We want each point to be hit only once. i.e. we want unique values (r,θ) for each (x,y).

$\Rightarrow 0 \le \theta < 2\pi$, $r \ge 0$.

So our region in (r,θ)-plane is a rectangle!



(points with same color coincide under T: we crush down r = 0 ; "unfold the fan")

$$\Rightarrow \iint_R f(x,y)\,dA = \int_0^{2\pi}\int_0^1 f(r,\theta)\cdot \underbrace{abr\,dr\,d\theta}_{dA \text{ from above.}}$$

If $f(x,y) = 1 \Rightarrow$ area $= \displaystyle\int_0^{2\pi}\int_0^1 ab\,r\,dr\,d\theta = \boxed{ab\pi}$

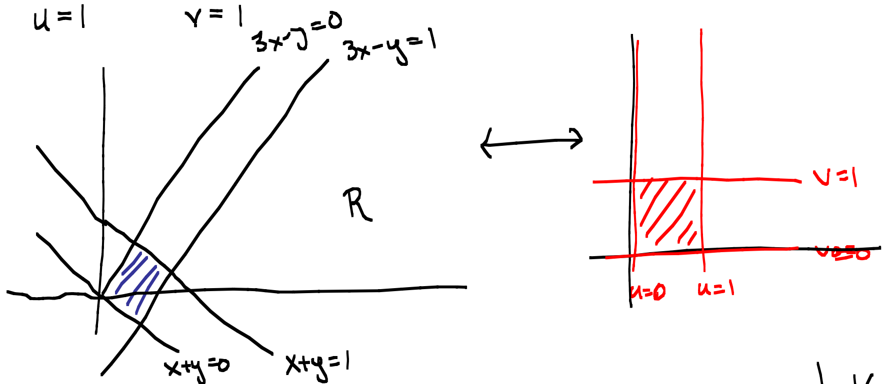
What are our steps to find the new region?

① Plug $x = x(u,v)$ and $y = y(u,v)$ into every boundary curve and look at the resulting region

② Remember constraints on (u,v) to get unique names.

Ex R: $3x-y=0$, $3x-y=1$, $x+y=0$, $x+y=1$ $\Big\}$ simplifies when $u=3x-y$, $v=x+y$

$\Rightarrow$ $u=0$, $u=1$, $v=0$, $v=1$



Now if $\left.\begin{array}{l} u=3x-y \\ v=x+y \end{array}\right\} \Rightarrow \left.\begin{array}{l} x=\frac{1}{4}(u+v) \\ y=\frac{3v-u}{4} \end{array}\right\} = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 1/4 & 1/4 \\ -1/4 & 3/4 \end{vmatrix} = 1/4$

Can use this to integrate things like $\frac{3x-y}{x+y+1}$

$$\iint_R \frac{3x-y}{x+y+1}\,dA = \int_0^1\int_0^1 \frac{u}{v+1}\cdot\frac{1}{4}\,du\,dv$$

$$= \frac{1}{4}\left(\frac{u^2}{2}\Big|_0^1\right)\cdot\Big(\ln(v+1)\Big)\Big|_0^1 = \boxed{\frac{1}{8}\ln 2}.$$

Last example: Set-up $\iint_R (x+y)e^{x^2-y^2}\,dA$ R bound by $\left\{\begin{array}{l} x+y=0 \\ x+y=1 \\ x-y=0 \\ x-y=1 \end{array}\right.$

Let $\left\{\begin{array}{l} u=x+y \\ v=x-y \end{array}\right\} \leftrightarrow \left\{\begin{array}{l} x=\frac{1}{2}(u+v) \\ y=\frac{1}{2}(u-v) \end{array}\right\}$

① dA: $\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{vmatrix} = -1/2 \Rightarrow dA = 1/2\,du\,dv$

② $\left.\begin{array}{lcl} x+y=0 & \leftrightarrow & u=0 \\ x+y=1 & \leftrightarrow & u=1 \\ x-y=0 & \leftrightarrow & v=0 \\ x-y=1 & \leftrightarrow & v=1 \end{array}\right\} =$ region: $0\le u\le 1$, $0\le v\le 1$

so $\iint_R (x+y)e^{x^2-y^2}\,dA = \int_0^1\int_0^1 ue^{uv}\,dv\,du.$ ← much easier!

How do we pick u & v?

1. Make the integrand as simple as possible $\left(\frac{3x-y}{x+y+1} \text{ or } (x+y)e^{x^2-y^2}, \text{etc}\right)$
2. If integrand isn't bad, then we can simplify the region $\left(\frac{x^2}{a^2}+\frac{y^2}{b^2} \le 1\right)$

Triple Integrals

Nothing really changes: $dA \rightsquigarrow dV$

1. Triple Integrals are iterated integrals
2. Fubini's Theorem holds (order doesn't matter)
3. Change of Var works analogously:

$$dV = \left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| du\,dv\,dw$$

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{array}{c|ccc|} & u & v & w \\ x & \partial x/\partial u & \partial x/\partial v & \partial x/\partial w \\ y & \partial y/\partial u & \partial y/\partial v & \partial y/\partial w \\ z & \partial z/\partial u & \partial z/\partial v & \partial z/\partial w \end{array}$$

Ex: R: $0 \le x \le 2$, $0 \le y \le 2$, $0 \le z \le 2$ $f(x,y,z) = xyz$

Then
$$\iiint_R f(x,y,z)\,dV = \int_0^2\int_0^2\int_0^2 xyz \overbrace{dx\,dy\,dz}^{\text{order does not matter}} = \int_0^2\int_0^2 \tfrac{1}{2}x^2yz\Big|_0^2 dy\,dz$$

$$= \int_0^2\int_0^2 2yz\,dy\,dz = \int_0^2 y^2z\Big|_0^2 dz =$$

$$\int_0^2 4z\,dz = 2z^2\Big|_0^2 = \boxed{8}$$