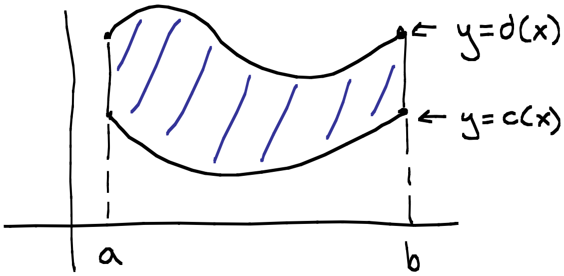
Lecture 15 - General & Polar Regions

Note Title

2 Big Cases:

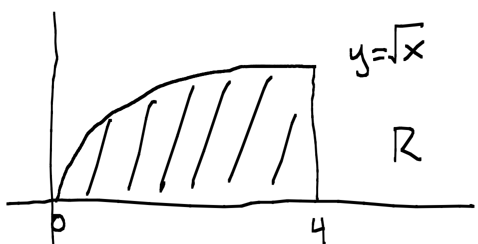
I. Bound by 2 functions of x: $a \le x \le b$, $c(x) \le y \le d(x)$



y a function of $x \Rightarrow$ integrate w.r.t. y first!

$$\iint_R F(x,y)\, dA = \int_a^b \int_{c(x)}^{d(x)} f(x,y)\, dy\, dx$$

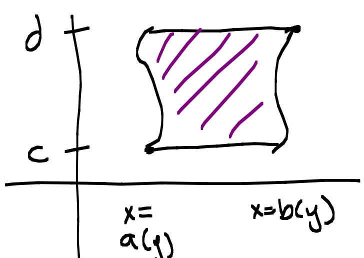
Ex:



$$\iint_R yx\, dA = \int_0^4 \int_0^{\sqrt{x}} yx\, dy\, dx$$

$$= \int_0^4 \frac{xy^2}{2}\Bigg|_{y=0}^{y=\sqrt{x}} dx = \int_0^4 \frac{x^2}{2}\, dx = \frac{x^3}{6}\Bigg|_0^4 = \boxed{\frac{32}{3}}$$

I. Bound by 2 functions of y:

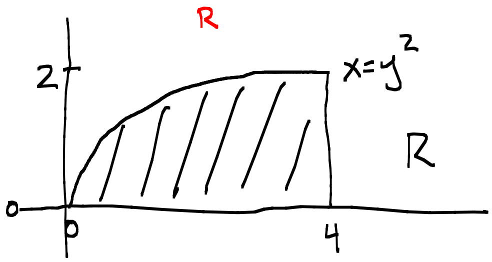


$c \le y \le d$
$a(y) \le x \le b(y)$

x a function of $y \Rightarrow$ integrate w.r.t. x first!

$$\iint_R F(x,y)\, dA = \int_c^d \int_{a(y)}^{b(y)} f(x,y)\, dx\, dy$$

Ex:



$$\iint_R yx\, dA = \int_0^2 \int_{y^2}^4 yx\, dy\, dx$$

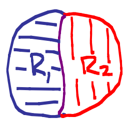
$$= \int_0^2 \frac{yx^2}{2}\Big|_{y^2}^{4} \, dy = \int_0^2 8y - \frac{y^5}{2} \, dy = \left(4y^2 - \frac{y^6}{12}\right)\Big|_0^2 = \boxed{32/3}$$

must be other answer because R and f are unchanged!

If R is both type I & type II, we get to choose which one to use.

If a region is neither type I nor type II, have to split things up.

Picture:



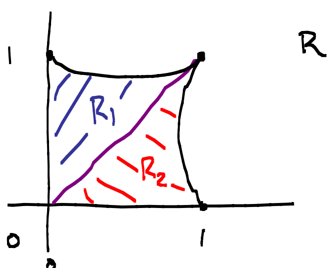
R_1 and R_2 intersect only along their boundary, and let R be all of the points in either R_1 or R_2.

$$\iint_R f(x,y)\, dA = \iint_{R_1} f(x,y)\, dA + \iint_{R_2} f(x,y)\, dA$$

Sometimes called "subadditivity". This means we can always break up a region into ones of Type I or II.

(Caveat: R_1 and R_2 must hit nicely ie only along boundary. This ensures that there is no volume from their intersection)

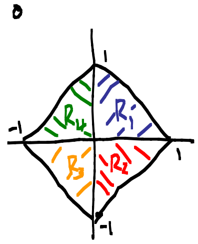
Ex:



Neither type I nor type II. But!

R_1 is type I. } Can use our previous
R_2 is type II. } discussion to find these

Ex:



R $(\Leftrightarrow \; |x|+|y| \le 1)$

Set-up $\iint_R 2 \, dA$:

R_1 : $0 \le x \le 1$, $0 \le y \le 1-x$
R_2 : $0 \le x \le 1$, $x-1 \le y \le 0$
R_3 : $-1 \le y \le 0$, $-1-y \le x \le 0$
R_4 : $0 \le y \le 1$, $y-1 \le x \le 0$

} all are Type I & II so any works

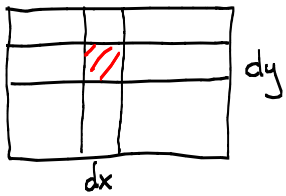
$$= \int_0^1 \int_0^{1-x} 2 \, dy\, dx + \int_0^1 \int_{x-1}^{0} 2 \, dy\, dx + \int_{-1}^{0} \int_{-1-y}^{0} 2 \, dx\, dy + \int_0^1 \int_{y-1}^{0} 2 \, dx\, dy$$

Double Integrals in Polar

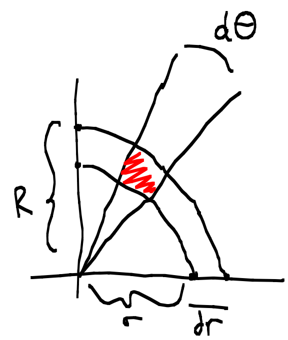
$x = r\cos\theta$
$y = r\sin\theta$

Find an expression for dA.

Rectangular:



$dA = dx\,dy$



Polar: $r = a$ is a circle
$\theta = \alpha$ is a ray $\Rightarrow$

dA = area of red piece.

Need: The area of a sector w/ angle θ and radius R is $\frac{1}{2}R^2\theta$

So red area: $\underset{\text{outer area}}{\left(\frac{1}{2}R^2\,d\theta\right)} - \underset{\text{inner area}}{\left(\frac{1}{2}r^2\,d\theta\right)} = \frac{1}{2}(R^2 - r^2)\,d\theta$

$= \frac{1}{2}(R+r)(R-r)\,d\theta$

$R - r = \Delta r \approx dr$, so if this is small, $R + r = (r + \Delta r) + r = 2r + \Delta r \approx 2r$

$\Rightarrow \Delta A \;\; \frac{1}{2}(2r)\,dr\,d\theta \Rightarrow$

$dA = r\,dr\,d\theta$