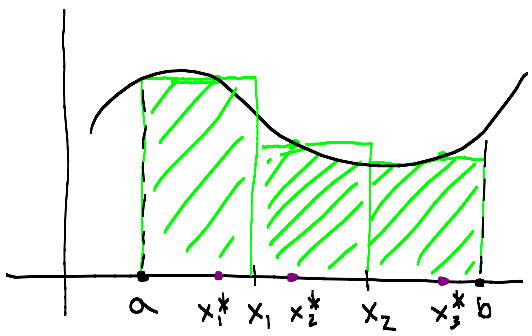
Lecture 14 – Multiple & Iterated Integrals

Note Title

Quick review of 1-var Riemann Sums:



To find area, subdivide $[a,b]$

$x_0 = a < x_1 < x_2 < \dots < x_{k-1} < b = x_k$

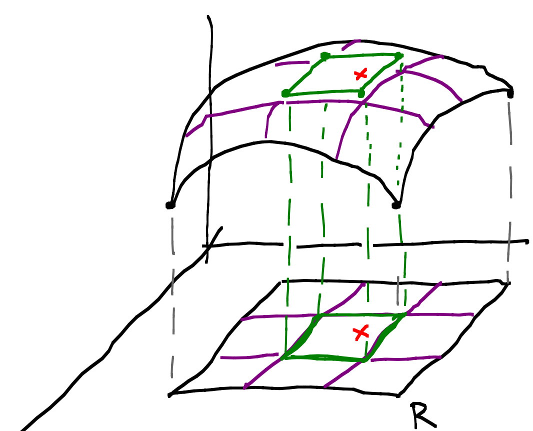
$\Delta x_i = x_i - x_{i-1}$

$x_{i-1} \le x_i^* \le x_i$

Then Area $\approx \displaystyle\sum_{i=1}^{k} f(x_i^*)\,\Delta x_i$ &

Def $\displaystyle\int_a^b f(x)\,dx = \lim_{\Delta x \to 0} \sum_{i=1}^{k} f(x_i^*)\,\Delta x_i.$

Now the 2 variable case: want to find volume under $z = f(x,y)$



Break $R = \{(x,y) \mid a \le x \le b,\ c \le y \le d\}$
$= [a,b] \times [c,d]$

into sub rectangles & look at a box over each rectangle of height $f(x_i^*, y_j^*)$.

Then volume is $\approx$

$$\sum f(x_i^*, y_j^*) \cdot \underbrace{\text{area of small rectangle}}_{\Delta A}$$

$$\Delta A = \underset{\uparrow \text{ x-subdivision}}{\Delta x_i} \cdot \underset{\uparrow \text{ y-subdivision}}{\Delta y_j}$$

Def $\displaystyle\iint_R f(x,y)\,dA = \lim_{\Delta A \to 0} \sum f(x_i^*, y_j^*)\,\Delta A$

a priori, this has nothing to do with 2 integrals. $\iint_R dA$ is a fixed symbol, defined by the Riemann sum.

$\Rightarrow$ very hard to compute

Ex $f(x,y) = 1$, $R = [0,1]\times[0,1]$ Then $\sum f(x_i^*, y_j^*)\,\Delta A = \sum \Delta A$

Now $\sum \Delta A$ = sum of areas of rectangles that cover $[0,1]\times[0,1]$

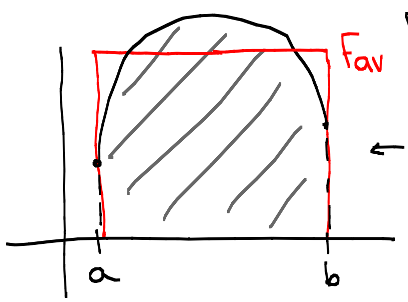
= area of $[0,1]\times[0,1] = 1$.

In fact, for any R,

$$\iint_R dA = \text{area}(R)$$

Can also look at average values:

f_{av} = value so that if we have one box with this height and base R, we get the right volume.



$\leftarrow$ 1 var case. So $f_{av}\cdot\text{area}(R) = \iint_R f(x,y)\,dA$.

We compute these with a "fundamental theorem".

Thm If f is continuous, then

$$\iint_R f(x,y)\,dA = \int_a^b\left(\int_c^d f(x,y)\,dy\right)dx$$

hold x constant

hold y constant

$$= \int_c^d\left(\int_a^b f(x,y)\,dx\right)dy$$

Ex: $R = [0,2]\times[0,2]$ $f(x,y) = xy$

$$\iint_R xy\,dA = \int_0^2 \underbrace{\int_0^2 xy\,dy}\,dx$$

$$\int_0^2 \left(\frac{xy^2}{2}\right)\Bigg|_{y=0}^{y=2} dx = \int_0^2 2x\,dx = \boxed{4}$$

Ex: $R = [0, \pi/2] \times [0, \pi/4]$ $\qquad f(x,y) = \sin x + \cos y$

$$\iint_R f(x,y)\, dA = \int_0^{\pi/2} \int_0^{\pi/4} \sin x + \cos y \; dy\, dx$$

$$\int_0^{\pi/2} \left(y \sin x + \sin y \right) \Big|_{y=0}^{y=\pi/4} dx = \int_0^{\pi/2} \frac{\pi}{4} \sin x + \frac{\sqrt{2}}{2} \; dx$$

$$= -\frac{\pi}{4} \cos x + x\frac{\sqrt{2}}{2} \Big]_{x=0}^{\pi/2} = \boxed{\frac{\pi}{4} + \frac{\pi\sqrt{2}}{4}}$$

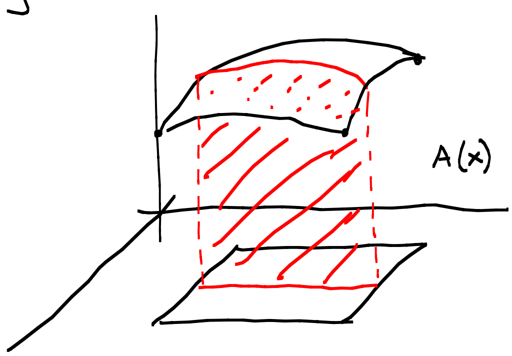
Fubini's Theorem If f is continuous, then

$$\int_a^b \int_c^d f(x,y)\, dy\, dx = \int_c^d \int_a^b f(x,y)\, dx\, dy.$$

In other words, the order of integration doesn't matter.

Why do we have such a formula? Compare with cross-sections.

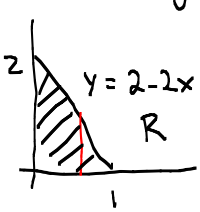
$A(x) = \int_c^d f(x,y)\, dy$ = area of cross-section $\parallel$ to (y,z)-plane @ x:



So formula for volume by cross-sectional area is

$$\iint_R f(x,y)\, dA = V = \int_a^b A(x)\, dx = \int_a^b \left(\int_c^d f(x,y)\, dy \right) dx$$

What about non-rectangular R? Express one variable's bounds in terms of the other:



$\}$ $f(x,y) = 3x$

So $0 \le x \le 1$ and for fixed x,

$0 \le y \le 2-2x$.

$$\text{So } \iint_R f(x,y)\, dA = \int_0^1 \int_0^{2-2x} 3x\, dy\, dx = \int_0^1 (3xy)\Big|_0^{2-2x} dx = \int_0^1 6x - 6x^2\, dx$$

$$= 3x^2 - 2x^3 \Big|_0^1 = \boxed{1}$$