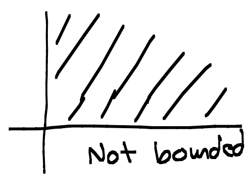
Lecture 13 – Extrema & Lagrange Multipliers

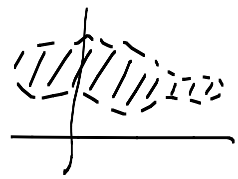
Note Title

Generalize 1-D notion: If f is continuous on $[a,b]$, then f has an absolute maximum and an absolute min.

We find this by finding crit points $c_1, \ldots$ and evaluating $f(a), f(b)$, and $f(c_1), \ldots$

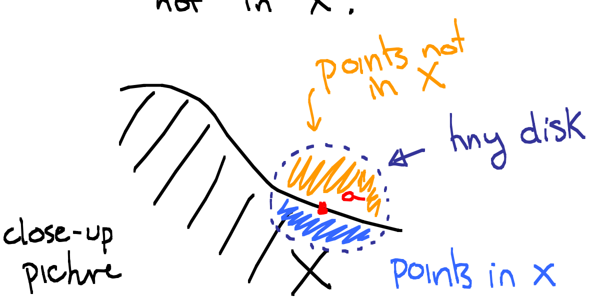
<u>Def</u>: A set X in $\mathbb{R}^2$ is <u>bounded</u> if it is contained in some big ball in the plane.





bounded

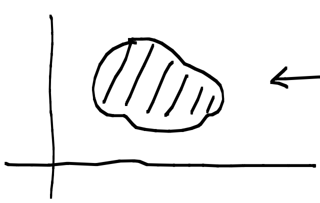
<u>Def</u> •) A point a is a <u>boundary point</u> of X if for all $r>0$, the open (punctured) disk $0 < d(x,a) < r$ has points in X and not in X.



•) The <u>boundary</u> of X is the set of all boundary points.

•) A set is <u>closed</u> if it contains all of its boundary points.

So



← closed

Not closed

The boundary is a curve (ie something 1-dim)

<u>Thm</u> If X is closed and bounded and if f is continuous, then f has an absolute maximum and an absolute minimum.

How do we find it?

① Find critical points ← easy!
② Find maxima & minima on the boundary

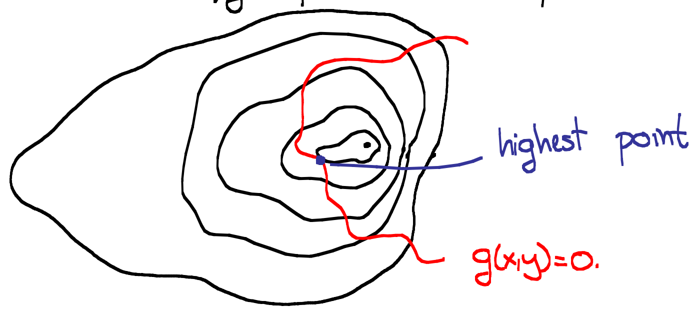
} Compare values

② is very hard in general $\rightsquigarrow$ Lagrange multipliers.

Goal: Understand how to maximize & minimize $f(x,y)$ subject to (x,y) on some curve: $g(x,y)=0$.

Moral idea: $z=f(x,y)$ describes the height of a mountain.

$g(x,y)=0$ describes a path in the (x,y) plane $\Rightarrow$ a path on the mountain. Topo map:



The maximum is the point on the path where the path is tangent to the level curve.

Why? If we aren't tangent then moving a small amount forward or backwards gives a change in height (the directional derivative is non-zero in the direction of the tangent vector). $\Rightarrow$ not at a max or min.

$\Rightarrow$ Maxima and minima occur when $\bar{\nabla} f = \mu \bar{\nabla} g$ some μ.

Method of Lagrange Multipliers:

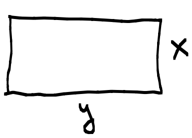
Find all (x,y) & μ s.t.

① $\bar{\nabla} f(x,y) = \mu \bar{\nabla} g(x,y)$

② $g(x,y)=0$

<u>Example</u> Maximize the area of a rectangular pen with perimeter 20.

important step! picture:



$f(x,y) = \text{area} = xy$

$g(x,y) = 2x+2y-20$ (perim = 20)

$$\left.\begin{array}{l}\bar{\nabla} f = \langle y, x\rangle \\ \bar{\nabla} g = \langle 2,2\rangle\end{array}\right\} \Rightarrow \left.\begin{array}{l} y = 2\mu \\ x = 2\mu \end{array}\right\} \Rightarrow x = y$$

$2x+2y-20=0$

$\Rightarrow 4x-20=0$

$\Rightarrow \boxed{x=y=5}$ (so $\mu = 5/2$) & max area $= \boxed{25}$.

Ex Extremize $f(x,y)=xy$ subject to $x^2+y^2=1$

Look for (x,y), μ s.t.

$$\begin{matrix} \nabla f = \mu \nabla g \\ x^2+y^2=1 \end{matrix} \quad \leftrightarrow \quad \left.\begin{matrix} y = 2\mu x \\ x = 2\mu y \end{matrix}\right\} \Rightarrow y = 2\mu(2\mu y)$$

$$x^2+y^2=1 \quad \rightarrow (2\mu y)^2+y^2=1$$

So $y(1-4\mu^2)=0$

$y^2(1+4\mu^2)=1$

If $y=0$, then $y^2(1+4\mu^2)=1$ fails

If $4\mu^2=1$, then $\mu=\pm 1/2$, & $2y^2=1 \Rightarrow y=\pm 1/\sqrt{2}$

$x = 2\mu y = \quad x = \pm 1/\sqrt{2}$

So look at the points $(x,y) = (1/\sqrt{2}, 1/\sqrt{2}), (-1/\sqrt{2}, 1/\sqrt{2}), (1/\sqrt{2}, -1/\sqrt{2}), (-1/\sqrt{2}, -1/\sqrt{2})$

$f(x,y) = \quad 1/2, \quad -1/2, \quad -1/2, \quad 1/2$

$\boxed{\text{maximum: } 1/2 \qquad \text{minimum: } -1/2}$

Very little changes with three variables:

To find extrema of $f(x,y,z)$ subject to $g(x,y,z)=0$

① Find all points (x,y,z) & λ s.t.

$$\nabla f(x,y,z) = \lambda \nabla g(x,y,z)$$
$$g(x,y,z)=0$$

② Plug-in these and look for biggest & smallest