Lecture 11 - Directional Derivatives

Note Title

Focus now on lines through a point and the resulting tangent vectors.

<u>Def</u> If $\vec{v} = \langle h, k \rangle$ and $f(x,y)$ is a differentiable function, then the <u>derivative of f at (a,b) in the direction of $\vec{v}$</u> is

$$D_{\vec{v}} f(a,b) = \lim_{t \to 0} \frac{f(a+ht, b+kt) - f(a,b)}{t}.$$

We normally restrict attention to unit vectors.

Know $D_{\vec{i}} f$ and $D_{\vec{j}} f$:

$$D_{\langle 1,0 \rangle} f = \frac{\partial f}{\partial x} \quad \& \quad D_{\langle 0,1 \rangle} f = \frac{\partial f}{\partial y}, \text{ by definition.}$$

<u>Ex</u>: $f(x,y) = x^2 + y^3$

$\vec{v} = \langle 1/\sqrt{2}, 1/\sqrt{2} \rangle$

$$D_{\vec{v}} f(x,y) = \lim_{t \to 0} \frac{f(x + t/\sqrt{2}, y + t/\sqrt{2}) - f(x,y)}{t}$$

$$= \lim_{t \to 0} \frac{\left(x^2 + \frac{2xt}{\sqrt{2}} + t^2/2\right) + \left(y^3 + 3y^2 t/\sqrt{2} + 3y\,t^2/2 + t^3/\sqrt{8}\right) - x^2 - y^3}{t}$$

$$= \lim_{t \to 0} \frac{2x}{\sqrt{2}} + t/2 + 3y^2/\sqrt{2} + 3yt/2 + t^2/\sqrt{8}$$

$$= \frac{2x}{\sqrt{2}} + \frac{3y^2}{\sqrt{2}}.$$

Can compute $D_{\vec{v}} f$ out of $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$ & $\vec{v} = \langle h, k \rangle$

$$D_{\vec{v}} f = \frac{\partial f}{\partial x} \cdot h + \frac{\partial f}{\partial y} \cdot k,$$

Proof is by the chain rule: $x = a + ht$, $y = b + kt$, so

$$\frac{df}{dt} = \frac{\partial f}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dt} = h \frac{\partial f}{\partial x} + k \frac{\partial f}{\partial y}.$$

<u>Ex</u>: $f(x,y) = x^6 + y^7$ (so the limit would be harder!)

$\vec{v} = \langle 1/\sqrt{5}, 2/\sqrt{5} \rangle$

$$D_{\vec{v}} f = \frac{6x^5}{\sqrt{5}} + \frac{7y^6 \cdot 2}{\sqrt{5}}.$$ Very easy!

The form of $D_{\bar{v}} f$ looks like a dot product, and this is easier to remember.

<u>Def</u> The gradient of $f(x,y)$ is

$$\bar{\nabla} f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle$$

If f is a function of 3 variables, then $\nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle$.

With this notation,

$$D_{\bar{v}} f = \bar{\nabla} f \cdot \bar{v}.$$

The gradient carries geometric meaning too:

$\bar{\nabla} f(a,b)$ points in the direction of steepest ascent.

$\longleftrightarrow$ if $\bar{u}$ is the unit vector in the dir of $\bar{\nabla} f$, then

$$D_{\bar{u}} f \geq D_{\bar{v}} f \quad \text{for all } \bar{v}.$$

<u>Why?</u> $D_{\bar{v}} f = \bar{\nabla} f \cdot \bar{v} = |\bar{\nabla} f| \cdot |\bar{v}| \cdot \cos\theta$

$= |\bar{\nabla} f| \cos\theta \leftarrow$ $\bar{v}$ is a unit vector.

$\cos\theta$ maximal if $\theta = 0$: $\bar{v}$ & $\bar{\nabla} f$ are in same dir.

Also see from geometry that $\bar{\nabla} f$ is $\perp$ to any level curves of f.

Moving along a level curve preserves the height $\Rightarrow D_{\bar{v}} f = 0 \Rightarrow \bar{\nabla} f \cdot \bar{v} = 0$,

$\bar{v}$ a tangent vector.

This is true in the level surface case.

level curve:	level surface:
$f(x,y) = c$	$f(x,y,z) = d$

<u>Thm</u> If $\bar{v}$ is a tangent vector to the surface $f(x,y,z) = d$ at (a,b,c), then $\bar{\nabla} f \cdot \bar{v} = 0$.

Tangent vector $\longleftrightarrow$ $\bar{r}'(t)$ for some $\bar{r}(t)$ a curve on the surface.

Apply $\frac{d}{dt}$ to $f(x,y,z) = d$:

$$\frac{d}{dt} f = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z} \cdot \frac{dz}{dt} = \bar{\nabla} f \cdot \bar{r}'(t)$$

$$\frac{d}{dt}(d) = 0 \qquad (d \text{ is constant})$$

So $\quad \vec{\nabla} f \cdot \vec{r}\,'(t) = 0.$

This means that $\vec{\nabla} f$ is a normal vector to the tangent plane (since, by the chain rule, $\vec{r}\,'(t)$ is in the tangent plane).

$\Rightarrow$

The equation of the tangent plane at (a,b,c) to $f(x,y,z) = d$ is

$$\vec{\nabla} f \cdot (\vec{r} - \vec{r}_0) = 0 \leftrightarrow \vec{\nabla} f \cdot \langle x-a,\, y-b,\, z-c \rangle = 0$$

$$\leftrightarrow \quad \frac{\partial f}{\partial x}(a,b,c) \cdot (x-a) + \frac{\partial f}{\partial y}(a,b,c) \cdot (y-b) + \frac{\partial f}{\partial z}(a,b,c) \cdot (z-c) = 0$$

Ex: $\quad \underbrace{x^2 + y^2 - z^2}_{f(x,y,z)} = 1 \quad @ \quad (2,1,2)$

$$\left.\begin{array}{lll} \partial & & @(2,1,2) \\ \frac{\partial f}{\partial x} = 2x & \rightsquigarrow & 4 \\ \frac{\partial f}{\partial y} = 2y & \longrightarrow & 2 \\ \frac{\partial f}{\partial z} = -2z & \rightsquigarrow & -4 \end{array}\right\} \Rightarrow \text{Tangent plane } @ (2,1,2): \quad 4(x-2) + 2(y-1) - 4(z-2) = 0$$

Have 2 equations for tangent planes.

$$z = g(x,y) \rightsquigarrow \quad z - g(a,b) = \frac{\partial g}{\partial x}(a,b)(x-a) + \frac{\partial g}{\partial y}(a,b)(y-b)$$

$$f(x,y,z) = d \rightsquigarrow \quad \frac{\partial f}{\partial x}(a,b,c) \cdot (x-a) + \frac{\partial f}{\partial y}(a,b,c) \cdot (y-b) + \frac{\partial f}{\partial z}(a,b,c) \cdot (z-c) = 0$$

Essentially the same thing: $\quad z = g(x,y) \leftrightarrow \underbrace{g(x,y) - z}_{} = 0$

So we get a level surface with: $\quad f(x,y,z) \quad \Rightarrow$

$$\frac{\partial f}{\partial x} = \frac{\partial g}{\partial x}, \quad \frac{\partial f}{\partial y} = \frac{\partial g}{\partial y}, \quad \frac{\partial f}{\partial z} = -1 \rightsquigarrow$$

$$\vec{\nabla} f \cdot (\vec{r} - \vec{r}_0) = \frac{\partial g}{\partial x} \cdot (x-a) + \frac{\partial g}{\partial y} \cdot (y-b) - (z - g(a,b)) = 0.$$

Remark: These level surfaces are the level sets for the $w = f(x,y,z)$ in $\mathbb{R}^4$.

The level surfaces are how this 3D object hits the space $w = d$.

i.e. This is a surface evolving through time.