MATH 231, Calculus III, Section 1, Fall 2007

Final Exam

Date: Dec. 11, Tuesday
Time: 9:00 am – 12:00 noon

Please write clearly, reduce answers to their simplest form, and box your answers.
To receive full credit you must show ALL your work.

Student's Name (Please print): Solutions

Pledge: On my honor as a student at the University of Virginia I have neither given nor received aid on this test.

Signature: ______________________________

Problem	**Points**	**Score**
1	15	
2	15	
3	20	
4	20	
5	25	
6	25	
7	20	
8	20	
9	10	
10	15	
11	15	
12	25	
13	25	
14	25	
15	25	
16 (Bonus)	25	
Total	**325/300**	

Problem 1 (15 points)
(a) (5 points) For what value(s) of t are $\mathbf{a} = \langle t+2, t, t\rangle$ and $\mathbf{b} = \langle t-2, t+1, 1\rangle$ orthogonal?

$$\bar{a}\cdot\bar{b} = \underbrace{(t^2-4) + (t^2+t) + t}_{2t^2+2t-4} = 0$$

$$2t^2+2t-4 = 2(t^2+t-2) \Rightarrow \boxed{t = 1, -2}$$

(b) (10 points) Consider the surface S given by $\rho^2 - 2\rho\cos\phi = 3$ in spherical coordinates. Convert the equation into an equation in rectangular coordinates and identify (describe) the surface.

$$\left.\begin{aligned}\rho^2 &= x^2+y^2+z^2\\ \rho\cos\phi &= z\end{aligned}\right\} \Rightarrow x^2+y^2+z^2-2z = 3$$

$+1 \qquad +1$

$$x^2+y^2+(z-1)^2 = 4 \longleftrightarrow$$ sphere of radius 2 centered at $(0,0,1)$.

Problem 2 (15 points)
Show that $\mathbf{F}$ is a conservative vector field, where
(a) (5 points) $\mathbf{F}(x,y) = (2+3x^2y)\mathbf{i} + (x^3-3y^2)\mathbf{j}$.

$$\frac{\partial f}{\partial x} = 2+3x^2y \Rightarrow f(x,y) = 2x + x^3y + g(y)$$

$$\frac{\partial f}{\partial y} = x^3 + g'(y) = x^3 - 3y^2 \Rightarrow g'(y) = -3y^2$$

$$g(y) = -y^3 + C$$

$$\Rightarrow \boxed{f(x,y) = 2x + x^3y - y^3 + C}$$

(b) (10 points) $\mathbf{F}(x,y,z) = (e^y+z^2)\mathbf{i} + xe^y\mathbf{j} + 2xz\mathbf{k}$.

$$\frac{\partial f}{\partial x} = e^y + z^2 \Rightarrow f = xe^y + xz^2 + g(y,z)$$

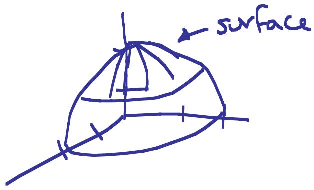
$$\frac{\partial f}{\partial y} = xe^y + \frac{\partial g}{\partial y} = xe^y \Rightarrow g(y,z) = h(z)$$

$$\frac{\partial f}{\partial z} = 2xz + h'(z) = 2xz \Rightarrow h(z) = C$$

$$\boxed{f = xe^y + xz^2 + C}$$

Problem 3 (20 points)

(a) (15 points) Sketch the surface S that is part of $x^2 + y^2 + z^2 = 4$ which lies above $z = 1$. Give the parametrization of S in terms of spherical angles ϕ and θ. Indicate the parameter domain of this parametrization.



$\bar{r} = \langle 2\cos\theta\sin\phi,\ 2\sin\theta\sin\phi,\ 2\cos\phi \rangle$

$0 \le \theta \le 2\pi$

$z \ge 1 \Rightarrow 2\cos\phi \ge 1 \Rightarrow \boxed{0 \le \phi \le \pi/3}$

(b) (5 points) Give the parametrization of S in terms of x and y. Indicate the parameter domain of this parametrization.

$z = \sqrt{4 - x^2 - y^2} \Rightarrow \boxed{\bar{r}(x,y) = \langle x, y, \sqrt{4-x^2-y^2} \rangle}$

$z \ge 1 \Rightarrow \sqrt{4-x^2-y^2} \ge 1 \Rightarrow 4 - x^2 - y^2 \ge 1 \Rightarrow \boxed{3 \ge x^2 + y^2}$

Problem 4 (20 points)

Find the directional derivative of $f(x, y) = \ln(x^2 + y^2)$ at the point $(2, 1)$ in the direction of $\mathbf{v} = \langle -1, 2 \rangle$.

$\frac{\partial f}{\partial x} = \frac{2x}{x^2 + y^2}$ $\qquad$ $\frac{\partial f}{\partial y} = \frac{2y}{x^2 + y^2}$

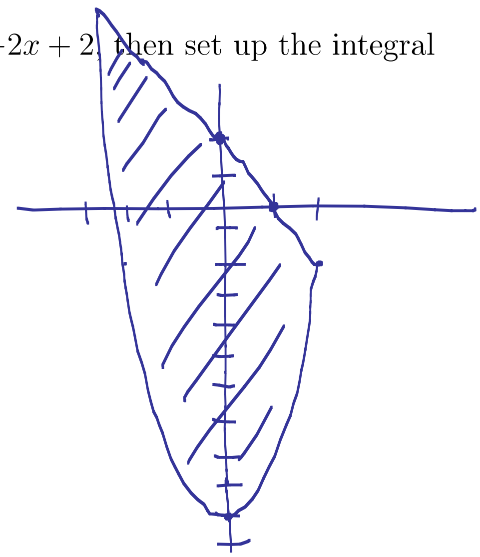
@ $(2,1)$: $\frac{4}{5}$ $\qquad$ $\frac{2}{5}$

$u = \left\langle \frac{-1}{\sqrt{5}}, \frac{2}{\sqrt{5}} \right\rangle \Rightarrow \boxed{\bar{\nabla} f(2,1) \cdot \bar{u} = 0.}$

Problem 5 (25 points)
Sketch the region D in $\mathbb{R}^2$ bounded by $y = 2x^2 - 10$ and $y = -2x + 2$, then set up the integral

$$I = \iint_D f(x, y)\, dA$$

in two different ways.



$$2x^2 - 10 = -2x + 2$$

$$x^2 + x - 6 = 0 \qquad (x+3)(x-2) = 0$$

Type I region:

$$-3 \le x \le 2$$

$$2x^2 - 10 \le y \le -2x + 2$$

Type II region:

$$-10 \le y \le -2 \qquad -2 \le y \le 8$$

$$-\sqrt{\frac{y+10}{2}} \le x \le \sqrt{\frac{y+10}{2}} \qquad -\sqrt{\frac{y+10}{2}} \le x \le \frac{y-2}{-2}$$

Problem 6 (25 points)
Evaluate the integral

$$I = \iint_R \frac{x+y}{x-y}\, dA$$

by making an appropriate change of variables, where R is the rectangle enclosed by $x+y=0$, $x+y=2$, $x-y=-3$ and $x-y=-1$.

$$\left.\begin{aligned} u &= x+y \\ v &= x-y \end{aligned}\right\} \Rightarrow \quad \begin{aligned} x &= \tfrac{1}{2}(u+v) \\ y &= \tfrac{1}{2}(u-v) \end{aligned} \quad \Rightarrow \quad J = \begin{vmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{vmatrix} = -1/2$$

bounds: $u=0$, $u=2$, $v=-3$, $v=-1$

So $$I = \int_0^2 \int_{-3}^{-1} \frac{u}{v} \cdot \frac{1}{2}\, dv\, du$$

$$= \left(\frac{u^2}{4} \Big|_0^2 \right) \left(\ln|v| \Big]_{-3}^{-1} \right)$$

$$= 1 \cdot \left(\ln|-1| - \ln|-3| \right)$$

$$= \boxed{-\ln 3}$$

Problem 7 (20 points)
Express the vector $\mathbf{b} = 2\mathbf{i} + 8\mathbf{j} - \mathbf{k}$ as the sum of two vectors, one parallel to $\mathbf{a} = \mathbf{i} + \mathbf{j} - 2\mathbf{k}$ and the other perpendicular to $\mathbf{a}$.

$$\mathrm{Proj}_{\bar{a}}\bar{b} = \frac{\bar{a}\cdot\bar{b}}{\bar{a}\cdot\bar{a}}\,\bar{a} = \frac{(2+8+2)}{(1+1+4)}\langle 1,1,-2\rangle$$

$$= \langle 2,2,-4\rangle \leftarrow \parallel \text{ to } \bar{a}$$

$$\mathrm{orth}_{\bar{a}}\bar{b} = \bar{b} - \langle 2,2,-4\rangle = \langle 0,6,3\rangle \leftarrow \perp \text{ to } \bar{a}.$$

Problem 8 (20 points)
Find the maximum and minimum values of the function $f(x, y, z) = 8x - 4z$ subject to the constraint $x^2 + 10y^2 + z^2 = 5$.

g $\quad$ f

$$\bar{\nabla} f = \lambda \bar{\nabla} g \qquad g = 5$$

$$8 = 2\lambda x \Rightarrow \lambda \neq 0$$
$$0 = 20\lambda y$$
$$-4 = 2\lambda z$$
$$\Rightarrow y = 0$$

So
$$\lambda \neq 0$$
$$\lambda x = 4$$
$$y = 0$$
$$\lambda z = -2 \Rightarrow \lambda = -2/z$$
$$x^2 + 10y^2 + z^2 = 5$$

$$(-2/z)x = 4 \Downarrow x = -2z$$

So $(-2z)^2 + z^2 = 5 \Rightarrow z = \pm 1 \Rightarrow x = \mp 2$

So max and min at
$$(2,0,-1) \xrightarrow{f} 12 \quad \text{max}$$
$$(-2,0,1) \xrightarrow{f} -12 \quad \text{min}$$

Problem 9 (10 points)
Find the velocity, acceleration and speed of the particle whose position function is given by $\mathbf{r}(t) = e^{2t}\,\mathbf{i} + \sqrt{t}\,\mathbf{j}$.

$\bar{r}'(t) = \bar{v}(t) = \langle 2e^{2t}, \frac{1}{2}t^{-1/2} \rangle \rightsquigarrow$ speed $= |\bar{v}| = \sqrt{4e^{4t} + \frac{1}{4t}}$

$\bar{v}'(t) = \bar{a}(t) = \langle 4e^{2t}, -\frac{1}{4}t^{-3/2} \rangle$

Problem 10 (15 points)
Find the limit, if it exists, or show that the limit does not exist:

$$L = \lim_{(x,y)\to(0,0)} \frac{x^2y^2}{x^2+y^2}.$$

If $(x,y) \neq (0,0)$, then $y^2 \leq x^2 + y^2 \neq 0 \Rightarrow \frac{y^2}{x^2+y^2} \leq 1$

so $\frac{x^2y^2}{x^2+y^2} \leq x^2 \leq x^2 + y^2$. $\Rightarrow$ If $\sqrt{x^2+y^2} < \delta$, then

$\frac{x^2y^2}{x^2+y^2} < \delta^2$, so choosing $\delta = \sqrt{\epsilon}$, then

$\left| \frac{x^2y^2}{x^2+y^2} - 0 \right| < \epsilon \Rightarrow L = 0.$

Problem 11 (15 points)
Find the (most general) potential function for $\mathbf{F}(x,y) = (-3\sin x + 2xy^2)\,\mathbf{i} + (x^2+1)2y\,\mathbf{j}$.

$\partial f/\partial x$ $\quad$ $\partial f/\partial y$

$f(x,y) = 3\cos x + x^2y^2 + g(y)$

$\frac{\partial f}{\partial y} = 2x^2y + g'(y) = 2x^2y + 2y \Rightarrow g'(y) = 2y$

$\Rightarrow g(y) = y^2 + C$

$\boxed{f(x,y) = 3\cos x + x^2y^2 + y^2 + C}$

Problem 12 (25 points)
Sketch the region E given by $x^2 + y^2 \le z \le 4$, then use the Divergence Theorem to evaluate

$$I = \iint_S \mathbf{F} \cdot d\mathbf{S},$$

where $\mathbf{F}(x, y, z) = y\mathbf{i} - x\mathbf{j} + z(x^2 + y^2)\mathbf{k}$ and S is the boundary of the region E.

Div Thm: $\quad I = \iiint_E \nabla \cdot \vec{F} \, dV = \iint_{x^2+y^2 \le 4} \int_{x^2+y^2}^{4} x^2+y^2 \, dz \, dA$

$\nabla \cdot \vec{F} = x^2+y^2$

$$= \int_0^{2\pi} \int_0^2 (4r^2 - r^4) \, r \, dr \, d\theta$$

$$= 2\pi \left(r^4 - \frac{r^6}{6} \Big|_0^2 \right) = 2\pi \left(16 - \frac{32}{3} \right)$$

$$= \boxed{\frac{32\pi}{3}}$$

Problem 13 (25 points)
Sketch the surface S that is part of $x^2 + y^2 = 4$ which lies between $z = 0$ and $z = 3$. Using a parametrization of S (specify a parameter domain), find an equation of the tangent plane to S at the point $(\sqrt{3}, 1, 2)$.

S is cyl parallel to z-axis through $x^2+y^2=4$

$x = \cos t$ $\quad 0 \le t < 2\pi$ $\quad \frac{\partial \bar{r}}{\partial t} = \langle -\sin t, \cos t, 0 \rangle$

$y = \sin t$ $\quad 0 \le z \le 3$

$z = z$ $\quad \frac{\partial \bar{r}}{\partial z} = \langle 0, 0, 1 \rangle$

$$\bar{n} = \frac{\partial \bar{r}}{\partial t} \times \frac{\partial \bar{r}}{\partial z} = \langle \cos t, \sin t, 0 \rangle$$

$\rightsquigarrow$ tangent plane $= \langle \sqrt{3}, 1, 0 \rangle \cdot \langle x - \sqrt{3}, y-1, z-2 \rangle = 0$

$$\sqrt{3} \cdot x - 3 + y - 1 = 0$$

$$\Rightarrow \boxed{x\sqrt{3} + y = 4}$$

Problem 14 (25 points)
Use Stokes' Theorem to evaluate

$$I = \iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S},$$

where $\mathbf{F}(x, y, z) = -yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k}$ and S is part of $x^2 + y^2 + z^2 = 9$ that lies inside $x^2 + y^2 = 4$ and above the xy-plane, with normal vector pointing in the positive z-direction.

$$\partial S = \left.\begin{array}{l} x^2+y^2+z^2=9 \; ; \quad x^2+y^2=4 \\ z \geq 0 \end{array}\right\} \Rightarrow \begin{array}{c} z=\sqrt{5} \\ x^2+y^2=4 \\ 0 \leq t \leq 2\pi \end{array} \rightsquigarrow \begin{array}{l} \bar{r}(t) = \langle 2\cos t, 2\sin t, \sqrt{5} \rangle \\ r'(t) = \langle -2\sin t, 2\cos t, 0 \rangle \end{array}$$

On Curve: $\bar{F} = \langle -2\sqrt{5}\sin t, 2\sqrt{5}\cos t, 4\cos t \sin t \rangle$

$$\bar{F} \cdot d\bar{r} = \left(4\sqrt{5}\sin^2 t\right) + \left(4\sqrt{5}\cos^2 t\right) + 0$$

$$= 4\sqrt{5}$$

Stokes: $\iint_S \bar{\nabla} \times \bar{F} \cdot d\bar{S} = \int_C \bar{F} \cdot d\bar{r} = \int_0^{2\pi} 4\sqrt{5}\, dt = \boxed{8\pi\sqrt{5}}$

Problem 15 (25 points)
Find the surface area of S, where S is a torus (doughnut) given by

$$\mathbf{r}(u,v) = \langle (2+\cos u)\cos v, (2+\cos u)\sin v, \sin u\rangle$$

with $0 \le u \le 2\pi$ and $0 \le v \le 2\pi$.

$$\frac{\partial r}{\partial u} = \langle \overset{\bar{i}}{-\sin u \cos v}, \overset{\bar{j}}{-\sin u \sin v}, \overset{\bar{k}}{\cos u} \rangle$$

$$\frac{\partial r}{\partial v} = \langle -(2+\cos u)\sin v, (2+\cos u)\cos v, 0 \rangle$$

$$\frac{\partial \bar{r}}{\partial u} \times \frac{\partial \bar{r}}{\partial v} = \langle -(2+\cos u)\cos u \cos v, (2+\cos u)\cos u \sin v, -(2+\cos u)\sin u \rangle$$

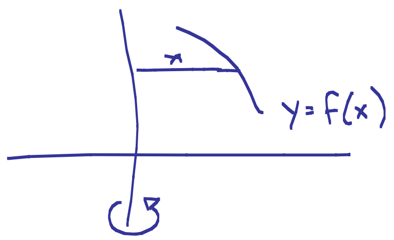
$$\left| \frac{\partial \bar{r}}{\partial u} \times \frac{\partial \bar{r}}{\partial v} \right| = (2+\cos u)\sqrt{\cos^2 u \cos^2 v + \cos^2 u \sin^2 v + \sin^2 u} = 2+\cos u$$

$$\Rightarrow dS = (2+\cos u)\, du\, dv$$

$$SA = \int_0^{2\pi}\int_0^{2\pi} 2+\cos u \; du\, dv = \boxed{8\pi^2}$$

Bonus Problem 16 (25 points)
Show that our definition of the surface area is consistent with the surface area formula from single-variable calculus.

Old SA formula: $SA = 2\pi \int_a^b x\sqrt{1+f'(x)^2}\,dx$



In our case, S is the surface

$z = f(r)$

So use cylindrical:

$$\bar{r}(r,\theta) = \langle r\cos\theta, r\sin\theta, f(r)\rangle$$

$$\frac{\partial \bar{r}}{\partial r} = \langle \overset{\bar{\imath}}{\cos\theta}, \overset{\bar{\jmath}}{\sin\theta}, \overset{\bar{k}}{f'(r)} \rangle$$

$$\frac{\partial \bar{r}}{\partial \theta} = \langle -r\sin\theta, r\cos\theta, 0\rangle$$

$$\underline{\quad} \times \underline{\quad} = r\langle -f'(r)\cos\theta, -f'(r)\sin\theta, 1\rangle$$

so $|\,\underline{\quad} \times \underline{\quad}\,| = r\sqrt{1+f'(r)^2}$

& $dS = |\,\underline{\quad} \times \underline{\quad}\,|\,dr\,d\theta$

So $$SA = \int_0^{2\pi}\int_a^b r\sqrt{1+f'(r)^2}\,dr\,d\theta$$

$$= 2\pi\int_a^b r\sqrt{1+f'(r)^2}\,dr. \quad \checkmark$$

(Could have param. everywhere. Important thing is that surf. of revolution play well with polar!)