Lecture 7 - More Matrix operations

Note Title 2/6/2008

Finished last time w/ Block matrices.

Ex: $A = \left[\begin{array}{c|c} 2 & 1 \\ -1 & 0 \\ \hline 3 & 1 \end{array}\right]$, $B = \left[\begin{array}{cc} 3 & 0 \\ \hline 2 & 1 \end{array}\right]$ $\quad AB = \left[\begin{array}{cc} 3\begin{bmatrix} 2 \\ -1 \end{bmatrix} + 2\begin{bmatrix} 1 \\ 0 \end{bmatrix} & 0\begin{bmatrix} 2 \\ -1 \end{bmatrix} + 1\begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ \hline 3\cdot 3 + 2\cdot 1 & 3\cdot 0 + 1\cdot 1 \end{array}\right] = \left[\begin{array}{cc} 8 & 1 \\ -3 & 0 \\ \hline 11 & 1 \end{array}\right]$

Very important example:

A, B: Don't partition A, portion B into its column vectors

$$A \cdot \left[\bar{b}_1 \,\middle|\, \cdots \,\middle|\, \bar{b}_n\right] = \left[A\bar{b}_1 \,\middle|\, \cdots \,\middle|\, A\bar{b}_n\right]$$

So we can also do matrix mult by $A\bar{b}_i$ for the columns of B.

Inhomogeneous systems:

Idea: to any system, we can associate a homogeneous system:

$$A\bar{x} = \bar{b} \Rightarrow A\bar{x} = \bar{0}.$$

Then any two solutions to $A\bar{x} = \bar{b}$ differ by a solution to $A\bar{x} = \bar{0}$:

$\bar{y}, \bar{z}$ solutions to $A\bar{y} = \bar{b} = A\bar{z} \rightsquigarrow A\bar{y} - A\bar{z} = \bar{b} - \bar{b} = \bar{0} \rightsquigarrow A(\bar{y} - \bar{z}) = \bar{0}$

$\rightsquigarrow \bar{y} - \bar{z}$ is a solution to $A\bar{x} = \bar{0}$.

So all solutions are of the form: $\bar{x}_h + \bar{x}_p$, $\bar{x}_h$ a sol to $A\bar{x}_h = \bar{0}$, $A\bar{x}_p = \bar{b}$.

$$\left.\begin{aligned} x + 2y + 3z &= 1 \\ 2y + 4z &= 4 \end{aligned}\right\} \Rightarrow \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c|c} 1 & 2 & 3 & 0 & 1 \\ 0 & 2 & 4 & 0 & 4 \end{array}\right] \rightsquigarrow \left[\begin{array}{ccc|c|c} 1 & 0 & -1 & 0 & -3 \\ 0 & 1 & 1 & 0 & 2 \end{array}\right] \rightsquigarrow \quad \bar{x} = \begin{bmatrix} s \\ -s \\ s \end{bmatrix} + \begin{bmatrix} -3 \\ 2 \\ 0 \end{bmatrix}$$

Now more operations: powers

$$A^k = \underbrace{A \cdot A \cdots A}_{k}$$

$\rightsquigarrow$ $\cdot)$ $A^k \cdot A^\ell = A^{k+\ell}$ $\quad$ Ex: $A = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$, $A^2 = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix} \rightsquigarrow A^4 = \begin{bmatrix} -4 & 0 \\ 0 & -4 \end{bmatrix}$

$\cdot)$ $(A^k)^\ell = A^{k\ell}$

$\cdot)$ $A^0 = I$

More, important operations:

Def If A is an $n \times m$ matrix, the transpose of A, A^t is the $m \times n$ matrix whose columns are the rows of A.

Ex: $A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{bmatrix}$, $A^t = \begin{bmatrix} 1 & 3 \\ 2 & 2 \\ 3 & 1 \end{bmatrix}$

Properties of t:

- $(A+B)^t = A^t + B^t$
- $(aB)^t = a \cdot B^t$
- $(AB)^t = B^t A^t \longleftarrow$ Swaps the order of multiplication!
- $(A^t)^t = A$

Def A matrix is symmetric if $A = A^t$

A matrix is antisymmetric if $A = -A^t$

Ex: $A = \begin{bmatrix} 1 & 3 \\ 5 & 7 \\ 9 & 11 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix} \rightsquigarrow AB = \begin{bmatrix} -6 & 2 \\ -14 & 10 \\ -22 & 18 \end{bmatrix} \rightsquigarrow (AB)^t = \begin{bmatrix} -6 & -14 & -22 \\ 2 & 10 & 18 \end{bmatrix}$

$$B^t A^t = \begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & 5 & 9 \\ 3 & 7 & 11 \end{bmatrix} = \begin{bmatrix} -6 & -14 & -22 \\ 2 & 10 & 18 \end{bmatrix}$$

One final definition:

Def If A is $n \times n$, the trace of A is the sum of the diagonal entries.

$$a_{1,1} + a_{2,2} + \ldots + a_{n,n}$$

$$\mathrm{Tr}\left(\begin{bmatrix} 3 & -2 \\ -2 & 2 \end{bmatrix} \right) = 3 + 2 = 5$$

Properties:

- $\mathrm{Tr}(A+B) = \mathrm{Tr}(A) + \mathrm{Tr}(B)$
- $\mathrm{Tr}(cA) = c\,\mathrm{Tr}(A)$
- $\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$
- $\mathrm{Tr}(A^t) = \mathrm{Tr}(A)$