Lecture 5 – Euclidean Geometry & Applications

Last time: finished with distance, getting last notion needed for ordinary geometry. Today we'll get some big geometry results.

<u>Thm</u> (Cauchy-Schwarz): $|\bar{u}\cdot\bar{v}| \le |\bar{u}|\cdot|\bar{v}|$ ← book writes this a $\|\bar{u}\|\cdot\|\bar{v}\|$

In $\mathbb{R}^1 = \mathbb{R}$, $\|\bar{v}\| = |\bar{v}|$, so Cauchy-Schwarz is obvious: $|\bar{u}\cdot\bar{v}| = |\bar{u}|\cdot|\bar{v}|$.

Just have to remember that vectors are a little badly behaved.

<u>Thm</u> 1) $\|\bar{u}+\bar{v}\| \le \|\bar{u}\| + \|\bar{v}\|$ (Triangle Inequality)

2) If $\bar{u}\cdot\bar{v} = 0$, then $\|\bar{u}+\bar{v}\|^2 = \|\bar{u}\|^2 + \|\bar{v}\|^2$

<u>Pf</u>: Look at $\|\bar{u}+\bar{v}\|^2 = (\bar{u}+\bar{v})\cdot(\bar{u}+\bar{v})$

F.O.I.L. the RHS: $= \bar{u}\cdot\bar{u} + \bar{u}\cdot\bar{v} + \bar{v}\cdot\bar{u} + \bar{v}\cdot\bar{v}$

$= \|\bar{u}\|^2 + 2\,\bar{u}\cdot\bar{v} + \|\bar{v}\|^2 \qquad (*)$

For part 1) take $|\cdot|$: $\le \|\bar{u}\|^2 + 2|\bar{u}\cdot\bar{v}| + \|\bar{v}\|^2$

(CS) $\le \|\bar{u}\|^2 + 2\|\bar{u}\|\cdot\|\bar{v}\| + \|\bar{v}\|^2$

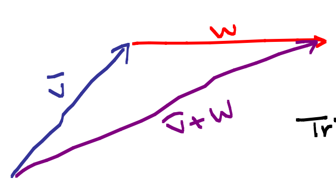
$= (\|\bar{u}\| + \|\bar{v}\|)^2$

Take $\sqrt{\ }$: $\|\bar{u}+\bar{v}\| \le \|\bar{u}\| + \|\bar{v}\|$

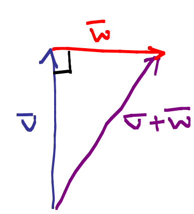
For part 2) If $\bar{u}\cdot\bar{v} = 0$, then the RHS of $(*)$ <u>is</u> $\|\bar{u}\|^2 + \|\bar{v}\|^2$, so

$(*)$ becomes $\|\bar{u}+\bar{v}\|^2 = \|\bar{u}\|^2 + \|\bar{v}\|^2$ ▨

<u>Pictures</u>



Triangle inequality says every side is at most the sum of the other sides



A few properties of $\|\cdot\|$ and $d(\cdot,\cdot)$:

$\|\bar{u}\| \ge 0$, $= 0 \leftrightarrow \bar{u} = 0$	$d(\bar{x},\bar{y}) \ge 0$, $= 0 \leftrightarrow x = y$
$\|c\bar{u}\| = |c|\cdot\|\bar{u}\|$	$d(\bar{x},\bar{y}) = d(\bar{y},\bar{x})$

So our notion of distance does capture our intuition, and we see how $\|\cdot\|$ works with the vector operations.

Applications!

I. Curve Fitting

II. Electrical Circuits

III. Traffic Flow

(II and III: Basically the same idea)

I. Know that given 2 points, there is a unique line between them. Can force this to have a coef of 1 for $y \Rightarrow$ unique equation.

This generalizes: If we have n points, there is a unique curve of the form

$$y = a_0 + a_1 x + \dots + a_{n-1}x^{n-1}$$

passing through them.

Find them using a linear systems:

$(-1, 2)$, $(1, 2)$, $(2, 4)$

$$\begin{aligned} 2 &= a_0 + a_1(-1) + a_2(-1)^2 \\ 2 &= a_0 + a_1(1) + a_2(1)^2 \\ 4 &= a_0 + a_1(2) + a_2(2^2) \end{aligned} \longleftrightarrow \left[\begin{array}{ccc|c} 1 & -1 & 1 & 2 \\ 1 & 1 & 1 & 2 \\ 1 & 2 & 4 & 4 \end{array}\right] \xrightarrow{-1} \left[\begin{array}{ccc|c} 1 & -1 & 1 & 2 \\ 0 & 2 & 0 & 0 \\ 0 & 3 & 3 & 2 \end{array}\right] \div 2$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2/3 \end{array}\right] \xrightarrow{-1} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 4/3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2/3 \end{array}\right] \rightsquigarrow y = 4/3 + (2/3)x^2$$

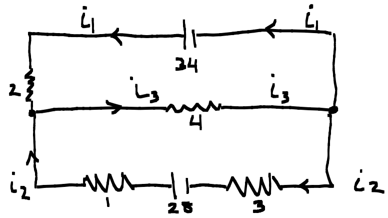
Moral Plug in every point!

II. Electrical Networks Circuit analysis

Kerchof's Laws

1) Junctions: "Conservation of current" - Current flowing in must flow out.

2) Paths: "Conservation of momentum" - Sum of IR along any path is the voltage.



$$i_1 + i_2 = i_3$$

$$2i_1 + 4i_3 = 34$$

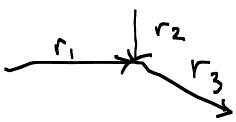
$$4i_2 + 4i_3 = 28$$

$$
\begin{bmatrix} 1 & 1 & -1 & 0 \\ 2 & 0 & 4 & 34 \\ 0 & 4 & 4 & 28 \end{bmatrix}
\begin{array}{l} \\ \div 2 \\ \div 4 \end{array}
\quad
\begin{bmatrix} 1 & 1 & -1 & 0 \\ 1 & 0 & 2 & 17 \\ 0 & 1 & 1 & 7 \end{bmatrix}
\Rightarrow
\begin{bmatrix} 1 & 1 & -1 & 0 \\ 0 & -1 & 3 & 17 \\ 0 & 1 & 1 & 7 \end{bmatrix}
\begin{array}{l} \\ \div -1 \\ \\ \end{array}
\quad
\begin{bmatrix} 1 & 1 & -1 & 0 \\ 0 & 1 & -3 & -17 \\ 0 & 0 & 4 & 24 \end{bmatrix}
\begin{array}{l} \\ \\ \div 4 \end{array}
$$

$$
\Rightarrow
\begin{bmatrix} 1 & 1 & -1 & 0 \\ 0 & 1 & -3 & -17 \\ 0 & 0 & 1 & 6 \end{bmatrix}
\Rightarrow
\begin{bmatrix} 1 & 1 & 0 & 6 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 6 \end{bmatrix}
\Rightarrow
\begin{bmatrix} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 6 \end{bmatrix}
$$

III. Traffic flow

1 rule: "rate of cars" is preserved, so cars go in and out at the same rate:



$\Rightarrow \quad r_3 = r_1 + r_2.$