Lecture 25 - Inner Product Spaces

Note Title 4/16/2008

In the previous lectures, we have seen how to bring $\mathbb{R}^n$ back into the picture:

- Picking a basis gives an identification $V \longleftrightarrow \mathbb{R}^n$
- Under this $(L: V \to W) \longleftrightarrow A \in M_{m \times n}$.

Today we will focus instead on how to import the geometry of $\mathbb{R}^n$ into the vector space story.

Def An inner product on V is a rule that assigns to each pair of vectors $(\bar{v}, \bar{w})$ a real number: $\langle \bar{v}, \bar{w} \rangle$ s.t.

1) $\langle a\bar{v}, \bar{w} \rangle = a \langle \bar{v}, \bar{w} \rangle$ "commutes w/ scalar mult"

2) $\langle \bar{u} + \bar{v}, \bar{w} \rangle = \langle \bar{u}, \bar{w} \rangle + \langle \bar{v}, \bar{w} \rangle$ "distributes over addition"

3) $\langle \bar{v}, \bar{v} \rangle \geq 0, \quad = 0 \longleftrightarrow \bar{v} = 0$ "possitive definite"

4) $\langle \bar{v}, \bar{w} \rangle = \langle \bar{w}, \bar{v} \rangle$ "symmetric"

V together with $\langle \, , \rangle$ is an inner product space.

In other words, we force all of the properties of the dot product to hold.

Remark 4) & 1)/2) imply $1')$: $\langle \bar{v}, a\bar{w} \rangle = a \langle \bar{v}, \bar{w} \rangle$ and

$2')$: $\langle \bar{w}, \bar{u} + \bar{v} \rangle = \langle \bar{w}, \bar{u} \rangle + \langle \bar{w}, \bar{v} \rangle$.

Ex 1) • on $\mathbb{R}^n$

2) on $P_n(x)$, have $\langle p, q \rangle = \int_a^b p(x) q(x) \, dx$ for any $b > a$.

$a = 0, b = 1$, $\quad \langle x, x^2 \rangle = \int_0^1 x^3 \, dx = 1/4$

3) on $C([a,b])$, $\quad \langle f, g \rangle = \int_a^b f(x) g(x) \, dx$.

4) On $P_2(x)$: $\quad \langle p, q \rangle = p(1)q(1) + p(2)q(2) + p(3)q(3)$.

All of our geometric notions carry through:

Def If V is an inner product space, then the length of $\bar{v}$ is

$$\| \bar{v} \| = \sqrt{\langle \bar{v}, \bar{v} \rangle}$$

So in $P_2(x)$ w/ $\int_0^1$, we have $\| x \| = \sqrt{\int_0^1 x^2 \, dx} = \frac{1}{\sqrt{3}}$

w/ other one: $\|x\| = \sqrt{1\cdot 1 + 2\cdot 2 + 3\cdot 3} = \sqrt{14}$

Thus we can talk about unit vectors: $\frac{x}{\|x\|}$, $\frac{\bar{v}}{\|\bar{v}\|}$, etc.

Def If $\bar{u}, \bar{v} \in V$, then the angle between $\bar{u}$ and $\bar{v}$ is given by

$$\cos\theta = \frac{\langle \bar{u}, \bar{v} \rangle}{\|\bar{u}\| \cdot \|\bar{v}\|} = \left\langle \frac{\bar{u}}{\|\bar{u}\|}, \frac{\bar{v}}{\|\bar{v}\|} \right\rangle .$$

Remark: That this is always $-1 \le$ " ≤ 1 is a consequence of the Cauchy-Schwartz theorem, proved exactly as for $\mathbb{R}^n$!

Def If $\langle \bar{u}, \bar{v} \rangle = 0$, say $\bar{u}$ and $\bar{v}$ are orthogonal: $\bar{u} \perp \bar{v}$

Ex: $P_2(x)$ w/ $\int_{-1}^{1}$: $\quad x \perp x^2$: $\int_{-1}^{1} x \cdot x^2 \, dx = \int_{-1}^{1} x^3 \, dx = 0.$

w/ other: $\langle x, x^2 \rangle = 1 + 2\cdot 4 + 3 \cdot 9 = 36 \neq 0$. So $x \not\perp x^2$.

Everything we learned about $\mathbb{R}^n$ w/ $\cdot$ and geometry holds here: get

- Projections
- Gram-Schmidt
- Approximations & Distance

This is very important when it comes to $C([a,b])$. Here we want to approximate a function by a simpler one (polynomial, trig, etc)

Def The distance between $\bar{u}$ and $\bar{v}$ is

$$d(\bar{u}, \bar{v}) = \|\bar{u} - \bar{v}\|.$$

If U is a subspace, then we can do something similar: Let $\text{Proj}_U \bar{v}$ be the projection onto U of $\bar{v}$. Then the distance from $\bar{v}$ to U is $\|\bar{v} - \text{Proj}_U \bar{v}\|$.

This is the smallest value of $\|\bar{v} - \bar{u}\|$ for any $\bar{u} \in U$.

In other words, $\text{Proj}_U \bar{v}$ is the vector in U that best approximates $\bar{v}$.

2 Examples:

1) Fourier Series: In $C([-\pi, \pi])$, consider the functions

$$\{1, \sin x, \cos x, \sin 2x, \ldots, \sin nx, \cos nx, \ldots\}$$

This is an orthogonal set (!) and

$$\langle 1, 1 \rangle = 2\pi, \quad \langle \sin nx, \sin nx \rangle = \langle \cos nx, \cos nx \rangle = \pi.$$

Since this set is orthogonal, it is easy to project onto the span of pieces.

Ex Approximate $f(x) = x$ by "trig polynomials" of "deg" ≤ 3

(things in the span) max n.

deg ≤ 3 = span$(1, \sin x, \cos x, \sin 2x, \cos 2x, \sin 3x, \cos 3x)$

have to find $\langle f(x), \text{"} \rangle$

$$\left\langle \begin{matrix} 1 \\ \cos x \\ \cos 2x \\ \cos 3x \end{matrix}, f(x) \right\rangle = 0, \text{ while}$$

$$\int_{-\pi}^{\pi} x \sin nx = -\frac{x \cos nx}{n} + \frac{\sin nx}{n^2}\Big]_{-\pi}^{\pi}$$

x	$\sin nx$
1	$-\frac{1}{n}\cos nx$ +
0	$-\frac{1}{n^2}\sin nx$ −

$$= -\frac{\pi}{n}(2\cos n\pi)$$

$$= \frac{(-1)^{n+1} 2\pi}{n}$$

$$\Rightarrow \quad x \approx \frac{\langle x, \sin x \rangle}{\langle \sin x, \sin x \rangle} \sin x + \cdots$$

$$= 2\sin x - \sin 2x + \frac{2}{3}\sin 3x$$