Lecture 22 - Coördinate Vectors

Today we start linking the abstract vector space story with our understanding of vectors and $\mathbb{R}^n$.

<u>Def</u> If $\mathcal{B} = \{\bar{v}_1, \dots, \bar{v}_n\}$ is a basis for V and $\bar{v} = a_1\bar{v}_1 + \dots + a_n\bar{v}_n$, then the <u>coordinate vector</u> of $\bar{v}$ w.r.t $\{\bar{v}_1, \dots, \bar{v}_n\}$ is the vector

$$[\bar{v}]_{\mathcal{B}} = \begin{bmatrix} a_1 \\ \vdots \\ \vdots \\ a_n \end{bmatrix}.$$

Conceptually, it is often helpful to label the rows with the elements of $\mathcal{B}$:

$$[\bar{v}]_{\mathcal{B}} = \begin{array}{c} \bar{v}_1 \\ \vdots \\ \bar{v}_n \end{array} \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}.$$

<u>Ex</u>: $\mathcal{E} = \{\bar{e}_1, \bar{e}_2\}$

$\mathcal{B} = \{(-1,1), (1,1)\}$

Then if $\bar{v} = (3,5)$, then

$$[\bar{v}]_{\mathcal{E}} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}, \quad [\bar{v}]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$$

<u>Ex</u>: $\mathcal{E} = \{1, x, x^2\}$

$\mathcal{B} = \{1, 1+x, 1+x+x^2\}$ in $P_2(x)$

Then if $p(x) = x^2 + 2x + 3$, then

$$[p]_{\mathcal{E}} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} \quad \text{while} \quad [p]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

By correctly picking our basis, we can try to make $[\bar{v}]$ as simple as possible. The choices of basis and the corresponding coordinates are closely related.

Prop Given 2 bases B, B' of V, there is a matrix ${}_{B'}C_B$ s.t. for all $\bar{v} \in V$,

$$[\bar{v}]_{B'} = {}_{B'}C_B\,[\bar{v}]_B.$$

The matrix ${}_{B'}C_B$ is easy to find: if $B = \{\bar{v}_1, \dots, \bar{v}_n\}$, then the j^{th} column of ${}_{B'}C_B$ is $[\bar{v}_j]_{B'}$.

Ex: $\mathcal{E} = \{1, x, x^2\}$ in $P_2(x)$, then

$B = \{1, x+1, x^2+x+1\}$

$$\left.\begin{aligned} 1 &= 1 + 0\cdot x + 0\cdot x^2 \\ x+1 &= 1 + 1\cdot x + 0\cdot x^2 \\ x^2+x+1 &= 1 + 1\cdot x + 1\cdot x^2 \end{aligned}\right\} \Rightarrow \quad {}_{\mathcal{E}}C_B = \begin{array}{c|ccc} & 1 & 1+x & 1+x+x^2 \\ \hline 1 & 1 & 1 & 1 \\ x & 0 & 1 & 1 \\ x^2 & 0 & 0 & 1 \end{array}$$

It's again helpful to add extra labels to make everything clearer: columns are labeled by source basis, rows by the target, just as in the above example.

Prop If B, B', B'' are bases of $\bar{v}$, then

$${}_{B''}C_B = {}_{B''}C_{B'} \cdot {}_{B'}C_B.$$

Cor For all B, B', ${}_{B'}C_B$ is invertible and $({}_{B'}C_B)^{-1} = {}_{B}C_{B'}$.

We need only see that ${}_BC_B = I$ for any basis B:

$B = \{\bar{b}_1, \dots, \bar{b}_n\}$, so $[\bar{b}_i]_B = \bar{e}_i$.

This all gives us a way to find ${}_{B'}C_B$: Let $\mathcal{E}$ be the standard basis

$$\begin{aligned} {}_{B'}C_B &= {}_{B'}C_{\mathcal{E}} \cdot {}_{\mathcal{E}}C_B \\ &= ({}_{\mathcal{E}}C_{B'})^{-1} \cdot {}_{\mathcal{E}}C_B \end{aligned}$$

In general, ${}_{\mathcal{E}}C_B$ is easy to compute: we express most vectors in terms of the standard basis, so we automatically know the columns.

Ex $V = P_2(x)$ $\quad \mathcal{E} = \{1, x, x^2\}$

$B = \{1, 1+x, 1+x+x^2\} \longrightarrow$

$$ {}_{\mathcal{E}}C_B = \begin{array}{c|ccc} & 1 & 1+x & 1+x+x^2 \\ \hline 1 & 1 & 1 & 1 \\ x & 0 & 1 & 1 \\ x^2 & 0 & 0 & 1 \end{array}$$

$C = \{1+x+x^2, x+x^2, x^2\}$

$$ {}_{\mathcal{E}}C_C = \begin{array}{c|ccc} & 1+x+x^2 & x+x^2 & x^2 \\ \hline 1 & 1 & 0 & 0 \\ x & 1 & 1 & 0 \\ x^2 & 1 & 1 & 1 \end{array}$$

$$\rightsquigarrow \quad {}_{e}C_{\mathcal{E}} = \left({}_{\mathcal{E}}C_{e}\right)^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \quad \Rightarrow \quad {}_{e}C_{B} = \begin{bmatrix} 1 & & \\ -1 & 1 & \\ & -1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 & 1 \\ & 1 & 1 \\ & & 1 \end{bmatrix}$$

$$= \begin{array}{c c} & \begin{array}{ccc} 1 & 1+x & 1+x+x^2 \end{array} \\ \begin{array}{c} 1+x+x^2 \\ x+x^2 \\ x^2 \end{array} & \begin{bmatrix} 1 & 1 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix} \end{array} .$$