# Lecture 2 - Gauss-Jordan Elimination

Note Title 1/21/2008

Ended last time with row-echelon form & reduced RE form. Today we will finish our analysis of systems via RE form, how to solve a system in RRE form, & how to reduce a matrix to RRE form.

## I. Number of solutions from RE form

Last time, saw 2 cases:

1) $\left[\begin{array}{ccc|c} 1 & * & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 1 & * \end{array}\right]$ has a unique solution.

2) $\left[\begin{array}{ccc|c} 1 & * & * & * \\ 0 & 0 & 1 & * \\ 0 & 0 & 0 & a \end{array}\right]$ has <u>no</u> solutions if $a \neq 0$. = "inconsistent"

Stand-in for any matrix in RE form with at least 1 zero on the diagonal.

3) Last case

$\left[\begin{array}{ccc|c} 1 & * & * & * \\ 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 \end{array}\right]$ ← can plug in <u>But</u> no info about y.
← can solve for z

infinite
This has $\infty$'ly many solutions. = "underdetermined" or "dependent"

<u>Moral</u>
- if there is a row like $[0 \dots 0 \mid 1]$, there are no solutions.
- If the RE form has all ones along the diagonal,
- If there are no bad rows, and any zeros on the diagonal, then $\infty$'ly many solutions.

Ex: Given the RE form, how many solutions?

a) $\left[\begin{array}{cccc|c} \underline{1} & 2 & 3 & 4 & 5 \\ 0 & \underline{1} & 2 & 3 & 4 \\ 0 & 0 & \underline{1} & 2 & 3 \\ 0 & 0 & 0 & \underline{1} & 2 \end{array}\right]$ ?

<u>One</u>

Diagonal is all ones.

b) $\left[\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 5 \\ 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 1 & 2 \\ \hline 0 & 0 & 0 & 0 & 1 \end{array}\right]$ ?

<u>None</u>

Row like $[0 \dots 0 \mid 1]$

c) $\left[\begin{array}{ccc|c} 1 & 2 & 3 & 17 \\ 0 & 0 & 1 & 23 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$ ?

<u>$\infty$'ly Many</u>

II. How do we solve when the system is in RRE form?

Ex: $\left[\begin{array}{ccc|c} 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right] \xleftrightarrow{\text{translate to eqns}}$

$$\begin{aligned} x + 2y &= 3 \leftarrow \text{doesn't depend on } z \\ z &= 2 \\ 0 &= 0 \end{aligned}$$

Can solve as

$$x = 3 - 2y$$
$$z = 2$$

or, calling $y$ a dummy variable like $s$,

$$\begin{aligned} x &= 3 - 2s \\ y &= s \\ z &= 2 \end{aligned}$$

; in vector form $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix} + \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} s$

What did we see?

2 Kinds of variables: • ones that can occur exactly once

= ones corresponding to the ones at the start of a row

• ones that could occur lots of times

= all the rest.

<u>Def</u> • A variable is a <u>lead</u> <u>variable</u> if it corresponds to a leading $\underline{1}$ in some row.

• A variable is a <u>free</u> <u>variable</u> if it is not a lead var.

(ie we can freely choose the values here)

<u>Moral</u> The reduced RE form tells us how to express lead variables in terms of free ones.

<u>Ex</u>
$$\left[\begin{array}{ccc|c} 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$
$\Rightarrow$ $x$ a lead var

$\Rightarrow$ $z$ a lead var

$\Rightarrow$ $y$ is free

<u>Ex</u>
$$\begin{array}{cccc} x & y & z & w \\ \end{array}$$
$$\left[\begin{array}{cccc|c} 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$
$\Rightarrow$ $y$ is lead

$\Rightarrow$ $z$ is lead

$\Rightarrow$ $x$ & $w$ are free

How do we solve?

① Give names to the values of free variables.

② Plug in those values to solve for lead variables.

<u>NB</u>: The names in ① <u>could</u> just be the variable name. Using a different one helps eliminate confusion.

Ex: $\left[\begin{array}{cccc|c} 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \longleftrightarrow \begin{array}{l} y+w=0 \\ z=1 \\ 0=0 \end{array}$

① $\begin{array}{l} x=a \\ w=b \end{array} \Rightarrow$ ② $\begin{array}{l} y+b=0 \\ z=1 \end{array} \Rightarrow \begin{array}{l} y=-b \\ z=1 \end{array}$

$\Rightarrow$ solution: $\begin{bmatrix} a \\ -b \\ 1 \\ b \end{bmatrix}$ for any choices of $a$ & $b$.

So with the RRE form, we can easily solve the system.

One other nice property: the RRE form is unique. ie it depends only on the system & each system has one.

III. Gauss-Jordan Elimination & Gaussian Elimination.

- GE: an algorithm for taking a matrix to RE form
- GJE: " " " " " " " RRE form.

Method is very simple. Describe it with a running example:

$$\left[\begin{array}{ccc|c} 2 & 4 & -2 & 6 \\ 1 & 2 & -1 & 3 \\ 3 & 6 & 2 & -1 \end{array}\right]$$

① Look at the $1^{st}$ non-zero column. Pick your favorite row w/ a non-zero entry in this column. If it is not the $1^{st}$ row, swap it with the $1^{st}$ row.

$$\left[\begin{array}{ccc|c} 2 & 4 & -2 & 6 \\ 1 & 2 & -1 & 3 \\ 3 & 6 & 2 & -1 \end{array}\right] \leftarrow \text{1 better than 2 or 3} \Rightarrow \left[\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 2 & 4 & -2 & 6 \\ 3 & 6 & 2 & -1 \end{array}\right]$$

② Scale the chosen row so that the leading term is 1.

③ Use row operations to kill off everything below the row in that col.

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 2 & 4 & -2 & 6 \\ 3 & 6 & 2 & -1 \end{array}\right] \begin{array}{c} -2 \\ + \\ -3,\ + \end{array} \Rightarrow \left[\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 5 & -10 \end{array}\right]$$

④ Holding the $1^{st}$ row fixed (ie ignoring the $1^{st}$ row and column), repeat with the remaining rows:

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 5 & -10 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 0 & 0 & 5 & -10 \\ 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{c} \\ \div 5 \\ \\ \end{array} \Rightarrow \left[\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

That's it. This gives Gaussian Elimination and a RE form.

for the RRE form, we add a step:

⑤ Use row operations to kill off everything above each leading 1.

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{c} + \\ \\ \\ \end{array} \Rightarrow \left[\begin{array}{ccc|c} 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

This extra step makes it Gauss-Jordan Elimination.

Let's do an example putting it all together:

$$\left[\begin{array}{cccc|c} 3 & 1 & 3 & -2 & -1 \\ -1 & -1 & -1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 6 \end{array}\right] \Rightarrow \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 6 \\ 3 & 1 & 3 & -2 & -1 \\ -1 & -1 & -1 & 1 & 0 \end{array}\right] \begin{array}{c} -3 \\ + \\ \\ \end{array} \Rightarrow \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 6 \\ 0 & -2 & 0 & -5 & -19 \\ 0 & 0 & 0 & 2 & 6 \end{array}\right] \begin{array}{c} \\ \div -2 \\ \div 2 \end{array}$$

$$\Rightarrow \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 6 \\ 0 & 1 & 0 & 5/2 & 19/2 \\ 0 & 0 & 0 & 1 & 3 \end{array}\right] \begin{array}{c} -1 \\ \\ -5/2 \end{array} \Rightarrow \left[\begin{array}{cccc|c} 1 & 1 & 1 & 0 & 3 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 & 3 \end{array}\right] \begin{array}{c} \\ -1 \\ \\ \end{array} \Rightarrow \left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 & 3 \end{array}\right]$$

$$= \left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 & 3 \end{array}\right]$$

$x, y, w$ lead variables

$z$ is a free variable

Let $z = s$.

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 & 3 \end{array}\right] \longleftrightarrow \begin{aligned} x + z &= 1 \\ y &= 2 \\ w &= 3 \end{aligned}$$

$\Rightarrow$ solutions are all of the form

$$\begin{aligned} x &= 1-s \\ y &= 2 \\ z &= s \\ w &= 3 \end{aligned} \longleftrightarrow \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 3 \end{bmatrix} + \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} s$$

basically the last col.

up to a sign on the 1st var, this is the 1st row.

Finally, all of this can be done in families.

Let $\begin{aligned} a_1x + b_1y + c_1z &= d_1 \\ &\vdots \\ a_nx + b_ny + c_nz &= d_n \end{aligned}$ , $\begin{aligned} a_1x + b_1y + c_1z &= e_1 \\ &\vdots \\ a_nx + b_ny + c_nz &= e_n \end{aligned}$ , ...

be a collection of systems of equations with identical left hand sides. Then we can solve them all at once by augmenting the coef matrix with all of the Right Hand Sides:

$$\left[\begin{array}{ccc|ccc} a_1 & b_1 & c_1 & d_1 & e_1 & \\ & \vdots & & \vdots & \vdots & \cdots \\ a_n & b_n & c_n & d_n & e_n & \end{array}\right]$$

Our matrix method makes this easy: just row reduce the left hand part and read out solutions.