Lecture 17 - Linear Independence & Basis

Note Title 3/24/2008

First some examples w/ span:

Ex $\{1, x+1, x^2+x+1\}$ spans $P_2(x)$.

We have to show that given $a_0+a_1x+a_2x^2$, there are a, b, c s.t.

$$a+b(x+1)+c(x^2+x+1) = a_0+a_1x+a_2x^2$$

$$\Updownarrow$$

$$\begin{aligned} a+b+c &= a_0 \\ b+c &= a_1 \\ c &= a_2 \end{aligned} \iff \left[\begin{array}{ccc|c} 1 & 1 & 1 & a_0 \\ 0 & 1 & 1 & a_1 \\ 0 & 0 & 1 & a_2 \end{array}\right]$$

Looking at the RE form, we see that there is a solution for any a_0, a_1, a_2!

Linear independence measures how many ways we can represent the same vector as a linear combination.

Def A set $\{\bar{v}_1, \dots, \bar{v}_n\}$ is linearly dependent if there are constants $a_1, \dots, a_n$, not all zero s.t.

$$a_1\bar{v}_1 + \dots + a_n\bar{v}_n = \bar{0} \leftarrow \text{linear dependence relation}$$

If a set is not lin. dep, then it is linearly independent.

Ex: In $M_2(\mathbb{R})$: $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \right\}$ is lin. ind:

$$a\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + b\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} + c\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} a+b+c & 2b \\ 0 & a+b-c \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{aligned} a+b+c &= 0 \\ 2b &= 0 \\ a+b-c &= 0 \end{aligned} \Rightarrow \begin{aligned} b&=0 \\ a+c&=0 \\ a-c&=0 \end{aligned} \Rightarrow a=b=c=0 \Rightarrow \text{independent.}$$

Do these also span $M_2(\mathbb{R})$? No! $\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \notin \text{Span}\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \right)$

This implies that $\left\{ \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \right\}$ is still linearly ind.

Span and independence are linked in a very key way:

1) If $\{\bar{v}_1, \dots, \bar{v}_n\}$ is lin. dependent, then for some i, $\bar{v}_i \in \text{Span}(\bar{v}_1, \dots, \bar{v}_{i-1}, \bar{v}_{i+1}, \dots, \bar{v}_n)$.

Equivalently, for some i, $\text{Span}(\bar{v}_1, \dots, \bar{v}_n) = \text{Span}(\bar{v}_1, \dots, \bar{v}_{i-1}, \bar{v}_{i+1}, \dots, \bar{v}_n)$.

2) If $\{\bar{v}_1,\dots,\bar{v}_n\}$ is lin independent, then $\mathrm{Span}(\bar{v}_1,\dots,\bar{v}_{i-1},\bar{v}_{i+1},\dots,\bar{v}_n) \subsetneq \mathrm{Span}(\bar{v}_1,\dots,\bar{v}_n)$ for all i.

So we can check independence by taking a non-zero vector in our set and adding in vectors one at a time, checking the span. We also see that subsets of linearly independent sets are lin. independent, just as supersets of spanning sets span. We want a way to split the difference.

Def A basis is a linearly independent, spanning set.

Ex: $\left\{ \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \right\}$ is a basis for $M_2(\mathbb{R})$

Now some big results:

Thm 1) Every vector space has a basis

2) All bases of a fixed vector space have the same # of elements.

Def The dimension of a vector space is the size of any basis.

This all together gives us rules for finding a basis:

I) If $\{\bar{v}_1,\dots,\bar{v}_n\}$ is linearly independent, then we can add in vectors 1 at a time, each time increasing the span until we have $\dim V$ vectors $\longleftrightarrow$ until adding new vectors doesn't change the span.

II) If $\{\bar{w}_1,\dots,\bar{w}_m\}$ spans, then we can throw out vectors 1 at a time, each time leaving the span unchanged until we have $\dim V$ vector $\longleftrightarrow$ until throwing out vectors drops the span.

Ex: $P_3(x)$. Start w/ a non-zero vector: 2.

$\mathrm{Span}(2) = \{a_0\}$, so take something outside of the span: $x+1$

$\mathrm{Span}(2, x+1) = \{a_0 + a_1 x\}$, so repeat: $x^3 + x^2$

$\mathrm{Span}(2, x+1, x^3+x^2) = \{a_0 + a_1 x + a_2 x^2 + a_2 x^3\}$, repeat: x^3.

$\mathrm{Span}(2, x+1, x^3+x^2, x^3) = P_3$ & $\{2, x+1, x^3+x^2, x^3\}$ is a basis.

P_n, $M_{n\times m}$ have standard bases:

P_n: $\{1, x, \dots, x^n\}$ & $\dim P_n = n+1$

M_2: $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$

$M_{n\times m}$: Let $E_{i,j}$ be the matrix that is 1 in position (i,j) and 0 otherwise.

Then $\{ E_{1,1}, \ldots, E_{1,m}, E_{2,1}, \ldots, E_{2,m}, \ldots, E_{n,1}, \ldots, E_{n,m} \}$ is a basis for $M_{n\times m}$ (these are basically the coordinate vectors).

$\dim M_{n\times m} = n \cdot m$.