Lecture 16 – Abstract Vector Spaces

Note Title 3/18/2008

Today we start the real heart of linear algebra: vector spaces.

Def A vector space is a set V together with an addition $(\bar{v},\bar{u}) \mapsto \bar{v}+\bar{u}$ and scalar multiplication $(a,\bar{v}) \longmapsto a\cdot\bar{v}$ satisfying

a1) $\bar{u}+(\bar{v}+\bar{w}) = (\bar{u}+\bar{v})+\bar{w}$

a2) there exists $\bar{0}$ s.t. $\bar{u}+\bar{0}=\bar{0}+\bar{u}=\bar{u}$

a3) for all $\bar{u}$, there exists $-\bar{u}$ s.t. $\bar{u}+(-\bar{u}) = (-\bar{u})+\bar{u} = \bar{0}$

a4) $\bar{u}+\bar{v} = \bar{v}+\bar{u}$

m1) $a\cdot(b\cdot\bar{v}) = (a\cdot b)\bar{v}$

m2) $1\cdot\bar{v} = \bar{v}$

m3) $(a+b)\cdot\bar{v} = a\cdot\bar{v}+b\cdot\bar{v}$

m4) $a\cdot(\bar{v}+\bar{w}) = a\cdot\bar{v}+a\cdot\bar{w}$

Ex: • $\mathbb{R}^n$ with usual $+,\cdot$ is a vector space.

• $M_{n,m}$ is a vector space w/ usual $+,\cdot$.

• $C([a,b]) = \{f:[a,b] \longrightarrow \mathbb{R} \mid f \text{ continuous}\}$

$$\left.\begin{aligned}(f+g)(x) &= f(x)+g(x)\\ (af)(x) &= a(f(x))\end{aligned}\right\}\ \text{pointwise operations}$$

forms a vector space:

a1) $(f+(g+h))(x) = f(x)+(g+h)(x) = f(x)+(g(x)+h(x)) = (f(x)+g(x))+h(x)$

$= (f+g)(x)+h(x) = ((f+g)+h)(x)$.

a2) $0(x)=0$ has $(0+f)=(f+0)=f$

a3) $-f = (-1)\cdot f$

a4) $(f+g)(x) = f(x)+g(x) = g(x)+f(x) = (g+f)(x)$

etc.

• Polynomials: $P(x) = \{\text{polynomials in } x\}$, usual $+,\cdot$

• Diff functions, etc, pointwise $+,\cdot$

Thm If V is a vector space, then

- $0 \cdot \bar{v} = \bar{0}$
- $a \cdot \bar{0} = \bar{0}$
- $(-1)\bar{v} = -\bar{v}$
- $a\bar{v} = \bar{0} \Rightarrow a = 0$ or $\bar{v} = \bar{0}$.

Def A subspace of a vector space is a (non-empty) subset U s.t.

$$\bar{u}, \bar{w} \in U \Rightarrow \bar{u} + \bar{w} \in U$$

$$\bar{u} \in U \Rightarrow a \cdot \bar{u} \in U.$$

Remark: It suffices to show that:

- $\bar{0} \in U$
- $\bar{u}, \bar{v} \in U$, then $a\bar{u} + b\bar{v} \in U$.

Ex: • $P(x)$ is a subspace of $C([a,b])$ for any a,b.

1) non-empty $(P(x) \ni 0(x))$

2) sum, scalar mult. of polys is a poly.

• $C'([a,b])$ is a subspace of $C([a,b])$.

$$C'([a,b]) = \{f : [a,b] \to \mathbb{R} \mid f' \text{ exists and is cont}\}$$

• In fact: $P(x) \subseteq C^{\omega}((a,b)) \subseteq C^{\infty}((a,b)) \subseteq \dots \subseteq C^1((a,b)) \subseteq C((a,b))$

• Let $P_n(x) = \{\text{polynomials of degree} \leq n\}$

ie $P_0(x) = \{a_0\}$
$P_1(x) = \{a_0 + a_1 x\}$
$\vdots$

Then $P_n(x)$ is a subspace of $P(x)$ and of $P_{n+1}(x)$.

Since $\deg 0 \leq n$ $\forall n \geq 0$, $0 \in P_n(x)$.

If $\underbrace{a_0 + \dots + a_n x^n}_{p}$ & $\underbrace{b_0 + \dots + b_n x^n}_{q} \in P_n(x)$, then

$$ap + bq = a(a_0 + \dots + a_n x^n) + b(b_0 + \dots + b_n x^n) = (aa_0 + bb_0) + \dots + (aa_n + bb_n)x^n \in P_n(x).$$

Our condition on subspaces had two parts: 1) have a distinguished vector & 2) contain lines/planes containing any two vectors. Can rephrase this.

Def $\bar{v} \in V$ is a linear combination of $\{\bar{v}_1, \dots, \bar{v}_n\}$ if $\bar{v} = a_1\bar{v}_1 + \dots + a_n\bar{v}_n$.

Def The span of $\{\bar{v}_1, \dots, \bar{v}_n\}$ is the set of all linear combinations of vectors in $\{\bar{v}_1, \dots, \bar{v}_n\}$: $\mathrm{Span}(\{\bar{v}_1, \dots, \bar{v}_n\})$ or $\mathrm{Span}(\bar{v}_1, \dots, \bar{v}_n)$

If $\mathrm{Span}(\bar{v}_1, \dots, \bar{v}_n) = V$, say $\{\bar{v}_1, \dots, \bar{v}_n\}$ spans V.

This is the same as for $\mathbb{R}^n$.

So a subspace of V is a non-empty subset U s.t. $\bar{u}, \bar{v} \in U \Rightarrow \mathrm{Span}(\bar{u}, \bar{v}) \subseteq U$.

Thm: $\mathrm{Span}(\bar{v}_1, \dots, \bar{v}_n)$ is always a subspace.

Pf 1) $\bar{0} = 0 \cdot \bar{v}_1 + \dots + 0 \cdot \bar{v}_n \in \mathrm{Span}(-)$.

2) If $\bar{v} = a_0 \bar{v}_1 + \dots + a_n \bar{v}_n$, $w = b_0 \bar{v}_1 + \dots + b_n \bar{v}_n$, then

$$a\bar{v} + b\bar{w} = a(a_0 \bar{v}_1 + \dots + a_n \bar{v}_n) + b(b_0 \bar{v}_1 + \dots + b_n \bar{v}_n) = (aa_0 + bb_0)\bar{v}_1 + \dots + (aa_n + bb_n)\bar{v}_n \in \mathrm{Span}(-)$$

So it's a subspace.

Note that the argument given was essentially the same as for P_n. This happens a lot, and $P_n = \mathrm{Span}(1, x, x^2, \dots, x^n)$.