Lecture 10 - Matrix Operations

Note Title 2/13/2008

Last time finished with rotations.

Recall: rotation through an angle ϕ:

$$T_\phi\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix}$$

<u>Ex</u>: Rotation through $\phi = \pi/2, \quad \pi/4, \quad \pi/6$:

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}, \quad \begin{bmatrix} \sqrt{3}/2 & -1/2 \\ 1/2 & \sqrt{3}/2 \end{bmatrix}$$

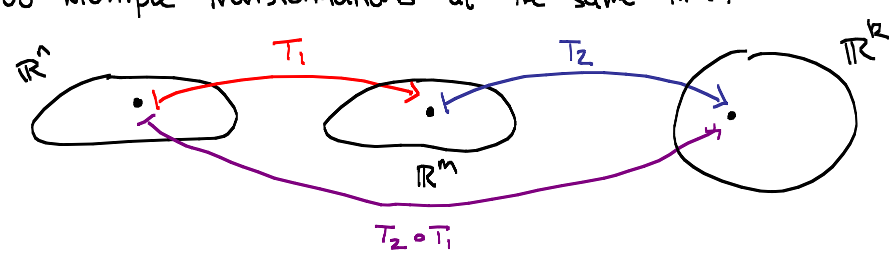
$$\begin{bmatrix} \sqrt{3}/2 & -1/2 \\ 1/2 & \sqrt{3}/2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \sqrt{3}/2 \\ 1/2 \end{bmatrix}$$

$$\text{''} \qquad \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1/2 \\ \sqrt{3}/2 \end{bmatrix}$$

These are all examples of "transformations": $\mathbb{R}^n \xrightarrow{T} \mathbb{R}^m$

$T(\bar{v}) = A\bar{v}$ also write $\bar{v} \longmapsto \underset{\uparrow\ \text{Image}}{A\bar{v}}$

Can do multiple transformations at the same time:



$$(T_2 \circ T_1)(\bar{v}) = T_2(T_1(\bar{v})) = A_2(A_1\bar{v}) = (A_2A_1)\bar{v}$$

$\rightsquigarrow$ Lets us combine: rotate, scale, reflect, etc.

$$T_{-\pi/4} = \begin{bmatrix} \sqrt{2}/2 & \sqrt{2}/2 \\ -\sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix}, \qquad R = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad T_{\pi/4} = \begin{bmatrix} \sqrt{2}/2 & -\sqrt{2}/2 \\ \sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix}$$

$$R \circ T_{-\pi/4} = \begin{bmatrix} \sqrt{2}/2 & \sqrt{2}/2 \\ \sqrt{2}/2 & -\sqrt{2}/2 \end{bmatrix} \Rightarrow T_{\pi/4} \circ R \circ T_{-\pi/4} = \begin{bmatrix} \sqrt{2}/2 & -\sqrt{2}/2 \\ \sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix} \begin{bmatrix} \sqrt{2}/2 & \sqrt{2}/2 \\ \sqrt{2}/2 & -\sqrt{2}/2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$=$ "reflect along $y = x$"

Another, hugely important family of transformations arise from "orthogonal matrices"

Def: A matrix A is orthogonal if

$$A^t = A^{-1}$$

Ex: $A = \begin{bmatrix} \sqrt{2}/2 & -\sqrt{2}/2 \\ \sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix}$

$$A^t = \begin{bmatrix} \sqrt{2}/2 & \sqrt{2}/2 \\ -\sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix} \Rightarrow A \cdot A^t = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = A^t \cdot A \Rightarrow A^{-1} = A^t.$$

In fact, every rotation is orthogonal:

$$\begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix} \cdot \begin{bmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{bmatrix} = \begin{bmatrix} \cos^2\phi + \sin^2\phi & \cos\phi\sin\phi - \sin\phi\cos\phi \\ \sin\phi\cos\phi - \sin\phi\cos\phi & \sin^2\phi + \cos^2\phi \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Why do we care?

If $\bar{u}$ and $\bar{v} \in \mathbb{R}^n$, then for all orthogonal matrices A,

$$\bar{u} \cdot \bar{v} = (A\bar{u}) \cdot (A\bar{v}).$$

$$\bar{u} \cdot \bar{v} = \bar{u}^t \cdot \bar{v} = \bar{u}^t \cdot (A^t \cdot A) \cdot \bar{v} = (\bar{u}^t \cdot A^t) \cdot (A \cdot \bar{v}) = (A \cdot \bar{u})^t \cdot (A \cdot \bar{v}) = (A\bar{u}) \cdot (A\bar{v})$$

So everything built out of the dot product is also preserved:

- length
- distance
- angles

All part of a more general set-up.

Def A linear transformation is a function $L: \mathbb{R}^n \to \mathbb{R}^m$ s.t.

$$L(\bar{u} + \bar{v}) = L(\bar{u}) + L(\bar{v}) \quad \& \quad L(c\bar{u}) = cL(\bar{u}).$$

Ex: $L\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 3x + y \\ 2y \end{bmatrix}$

$$L\left(\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} w \\ z \end{bmatrix}\right) = L\left(\begin{bmatrix} x + w \\ y + z \end{bmatrix}\right) = \begin{bmatrix} 3(x+w) + (y+z) \\ 2(y+z) \end{bmatrix} = \begin{bmatrix} (3x+y) + (3w+z) \\ (2y) + (2z) \end{bmatrix} = \begin{bmatrix} 3x+y \\ 2y \end{bmatrix} + \begin{bmatrix} 3w+z \\ 2z \end{bmatrix} = L\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) + L\left(\begin{bmatrix} w \\ z \end{bmatrix}\right).$$

$$L\left(c\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 3cx + cy \\ 2cy \end{bmatrix} = \begin{bmatrix} c(3x+y) \\ c(2y) \end{bmatrix} = cL\left(\begin{bmatrix} x \\ y \end{bmatrix}\right).$$

Notice $L(\bar{v}) = \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix} \bar{v}$

Any function $\mathbb{R}^n \to \mathbb{R}^m$ is determined by its coordinates. It is linear if every coordinate is a linear expression.

Ex: $L(x,y) = (7x, 18y)$ — linear

$L(x,y) = (e^x, y)$ — not linear

Check like with the previous example.

Thm: For every linear transformation $L: \mathbb{R}^n \to \mathbb{R}^m$, there is an $m\times n$ matrix A such that

$$L(\bar{v}) = A\bar{v}$$

How to find A: The i^{th} column of A, $\bar{a}_i$, is $L(\bar{e}_i)$.

Ex: $L\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x+2y \\ 3x-4y \end{bmatrix}$.

$$\left.\begin{array}{l} 1)\ L\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 3 \end{bmatrix} \\ \\ 2)\ L\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ -4 \end{bmatrix} \end{array}\right\} \Rightarrow A = \begin{bmatrix} 1 & 2 \\ 3 & -4 \end{bmatrix}$$

$\checkmark$: $\begin{bmatrix} 1 & 2 \\ 3 & -4 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x+2y \\ 3x-4y \end{bmatrix} = L\left(\begin{bmatrix} x \\ y \end{bmatrix}\right)$.