On the non-existence of elements of Kervaire invariant one

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Poincaré Conjecture & Milnor's Question

Milnor's Questions

How many smooth structures are there on the n-sphere?

Theorem (Poincaré Conjecture: Smale-Freedman-Perelman)

If M is a homotopy n-sphere that is a manifold, then M is homeomorphic to S^n.

Kervaire-Milnor 1963

Definition

Let Θ_n be the group of h-cobordism classes of homotopy n-spheres with addition connect sum.

ψ_n

Have a map

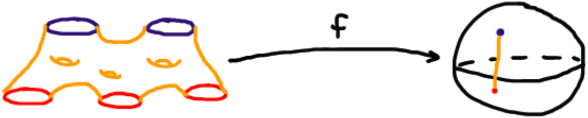
$$\Theta_n \xrightarrow{\psi_n} \pi_n^s / Im(J).$$

Pontryagin's Work

Definition

A framed n-manifold is an n-manifold with a continuous choice of basis for the normal vectors at every point

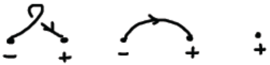
$$\{M^n \subset \mathbb{R}^{n+k}\}/\text{cobordism} \longleftrightarrow \pi_{n+k} S^k$$



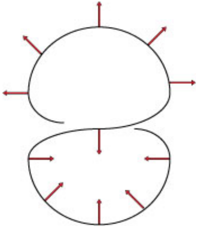
$$\text{Framings on } S^k \longleftrightarrow Im(J)$$

Pontryagin's Computations

$\pi_0^s = \mathbb{Z}$:



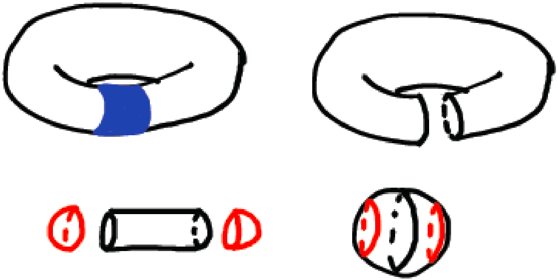
$\pi_1^s = \mathbb{Z}/2\mathbb{Z}$:



Framed Surgery

π_2^s:

Pontryagin: **framed surgery**



Consequences

ψ_2 not onto

The map

$$\Theta_2 \to \pi_2^s / Im(J)$$

is not surjective.

Get a map

$$\mu \colon H_n(M; \mathbb{Z}) \to \mathbb{Z}/2\mathbb{Z}.$$

If we can do surgery: 0, if we can't: 1.

Back to Ψ_n

Definition

Let bP_{n+1} be the subset of Θ_n of those spheres that bound parallelizable (frameable) manifolds.

Theorem (Kervaire-Milnor)

If $n \not\equiv 2 \mod 4$, then there is an exact sequence

$$0 \to bP_{n+1} \to \Theta_n \xrightarrow{\Psi_n} \pi_n^s/Im(J) \to 0.$$

If $n \equiv 2 \mod 4$, then there is an exact sequence

$$0 \to bP_{n+1} \to \Theta_n \xrightarrow{\Psi_n} \pi_n^s/Im(J) \xrightarrow{\Phi_n} \mathbb{Z}/2 \to bP_n \to 0.$$

bP_{n+1}

bP_{n+1} has a simple structure: it's finite cyclic!

Theorem (Kervaire-Milnor)

$$|bP_{n+1}| = \begin{cases} 1 & n \equiv 0 \mod 2 \\ 1 \text{ or } 2 & n \equiv 1 \mod 4 \\ 2^{2k-2}(2^{2k-1}-1)num\left(\frac{4B_k}{k}\right) & n = 4k-1 > 3. \end{cases}$$

Theorem (Adams, Mahowald)

$$|Im(J)| = \begin{cases} 1 & n \equiv 2,4,5,6 \mod 8, \\ 2 & n \equiv 0,1 \mod 8, \\ denom\left(\frac{B_k}{4k}\right) & n = 4k-1. \end{cases}$$

Kervaire Problem

Definition (Kervaire Invariant)

If M is a framed $(4k+2)$-manifold, then the Kervaire invariant Φ_{4k+2} is the obstruction to surgery in the middle dimension.

Kervaire Invariant One Problem

Is there a smooth n-manifold of Kervaire invariant one?

Adams Spectral Sequence

Adams Spectral Sequence

There is a spectral sequence with

$$E_2 = \mathrm{Ext}_{\mathcal{A}}\big(H^*(Y), H^*(X)\big)$$

and converging to $[X, Y]$.

- (Adem) $\mathrm{Ext}^1(\mathbb{F}_2, \mathbb{F}_2)$ is generated by classes h_i, $i \geq 0$.
- h_j survives the Adams SS if $\mathbb{R}^{2^j}$ admits a division algebra structure:

$$d_2(h_j) = h_0 h_{j-1}^2.$$

Browder's Reformulation

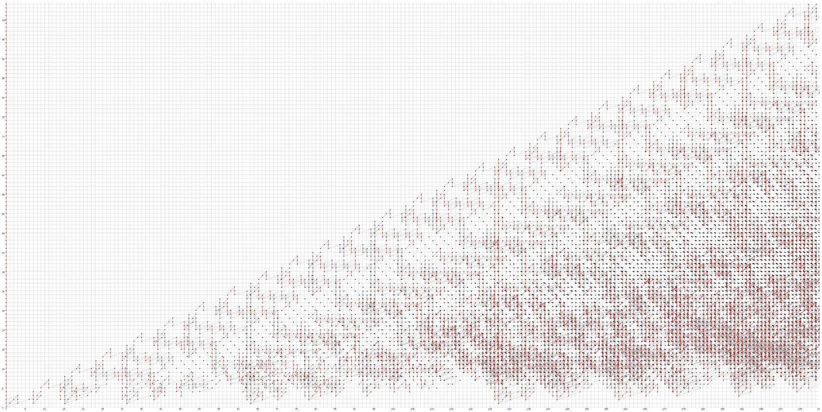
Theorem (Browder 1969)

1. *There are no smooth Kervaire invariant one manifolds in dimensions not of the form* $2^{j+1} - 2$.
2. *There is such a manifold in dimension* $2^{j+1} - 2$ *iff* h_j^2 *survives the Adams spectral sequence.*

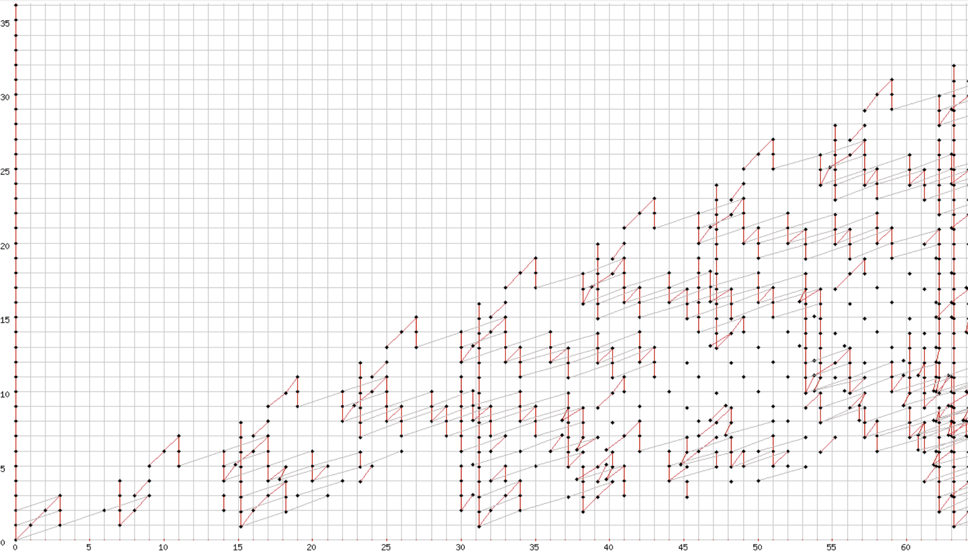
Classical Examples

h_1^2: $S(\mathbb{C}) \times S(\mathbb{C})$ h_2^2: $S(\mathbb{H}) \times S(\mathbb{H})$ h_3^2: $S(\mathbb{O}) \times S(\mathbb{O})$

Adams Spectral Sequence



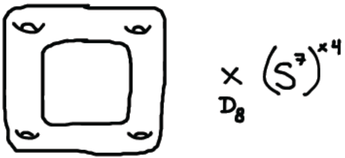
Adams Spectral Sequence



Previous Progress

Theorem (Mahowald-Tangora)

The class h_4^2 survives the Adams SS.



Theorem (Barratt-Jones-Mahowald)

The class h_5^2 survives the Adams SS.

Main Theorem

Theorem (H.-Hopkins-Ravenel)

For $j \geq 7$, h_j^2 does not survive the Adams SS.

We produce a cohomology theory $\Omega^*(-)$ such that

1. the cohomology theory detects the Kervaire classes,
2. $\Omega^{-2}(pt) = 0$, and
3. $\Omega^{k+256}(X) \cong \Omega^k(X)$.

We rigidify to a C_8-equivariant spectrum $\Omega_{\mathbb{O}}$:

$$\Omega = \Omega_{\mathbb{O}}^{C_8} \simeq \Omega_{\mathbb{O}}^{hC_8}.$$

Cohomology Theories

Cohomology Theory

$$\{\text{Topological Spaces}\} \xrightarrow{E^*} \{\text{Graded Abelian Groups}\}$$

satisfying

1. Homotopy Invariance: $f \simeq g \Rightarrow E^*(f) = E^*(g)$
2. Excision: $X = B \cup_A C$, then have a long exact sequence

$$\cdots \to E^n(X) \to E^n(C) \oplus E^n(B) \to E^n(A) \to E^{n+1}(X) \to \ldots$$

Example

1. Singular cohomology
2. K-theory (vector bundles on X)

Spectra in Algebraic Topology

Idea

Spectra represent cohomology theories: $E^n(X) = [X, E_n]$

Spectrum

A sequence of spaces $E_1, E_2, \ldots$ together with equivalences

$$E_n \cong \Omega E_{n+1} = Maps(S^1, E_{n+1})$$

1. Singular homology: $H\mathbb{Z}_n = K(\mathbb{Z}, n)$
2. K-Theory: $KU_{2n} = \mathbb{Z} \times BU$, $KU_{2n-1} = U$.

Equivariant Homotopy

Equivariant Homotopy

Homotopy theory for spaces with a G-action.

- For $H \subset G$, have "fixed points" E^H.
- There are spheres for every real representation.

Example

If $G = \mathbb{Z}/2$, then we have $S^{\rho_2} = \mathbb{C}^+$ and S^2.

$$\left(S^{\rho_2}\right)^{\{e\}} = S^2 \quad \left(S^{\rho_2}\right)^{C_2} = \mathbb{R}^+ = S^1.$$

What is our cohomology theory?

1. Begin with the bordism theory for (almost) complex manifolds: MU.

Theorem

For a finite index subgroup $H \subset G$, there is a multiplicative functor
$$N_H^G \colon H\text{–}Spectra \to G\text{–}Spectra.$$

$$MU^{(C_8)} = N_{C_2}^{C_8} MU\colon \quad \begin{tikzcd} \mathrm{MU} \arrow[rr, bend right] & \otimes & \mathrm{MU} \arrow[rr, bend right] & \otimes & \mathrm{MU} \arrow[rr, bend right] & \otimes & \mathrm{MU} \arrow[llllll, bend right, "\overline{(-)}"'] \end{tikzcd}$$

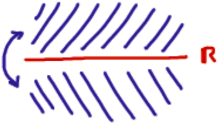
2. Localize: $\Omega_{\mathbb{O}} = \bar{\Delta}^{-1} MU^{(C_8)}$.

Structure of $MU^{(C_8)}$

Goal

Want to compute the homotopy groups of the fixed points for spectra like $\Omega_{\mathbb{O}}$.

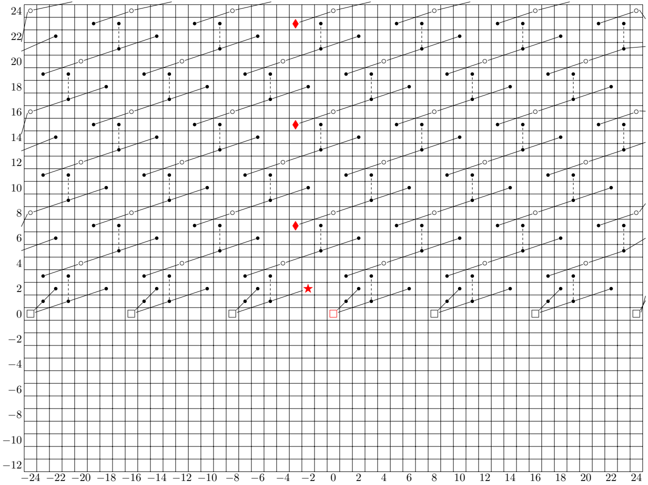
Start with Schubert cells: The Grassmanians $Gr_n(\mathbb{C}^k)$ all have cells of the form $\mathbb{C}^m$



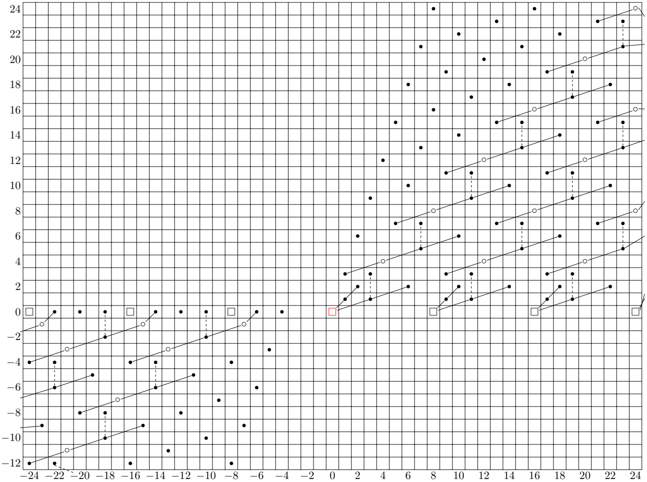
For $MU^{(C_8)}$, we therefore see three kinds of representation spheres:

1. $S^{k\rho_8}$
2. $Ind_{C_4}^{C_8} S^{k\rho_4}$
3. $Ind_{C_2}^{C_8} S^{k\rho_2}$

Advantages of the Slice SS



Advantages of the Slice SS



Slice Filtration of $MU^{(C_8)}$

Theorem

There is a multiplicative filtration of $MU^{(C_8)}$ with associated graded

$$\bigvee_{p\in\mathcal{P}} Ind_{H(p)}^{C_8} S^{k(p)\rho_{H(p)}} \wedge H\underline{\mathbb{Z}}.$$

Corollary

There is a spectral sequence

$$E_2^{s,t} = \bigoplus_{\substack{p\in\mathcal{P} \\ |p|=t}} H_{t-s}^{H(p)}\left(S^{k(p)\rho_{H(p)}+V}; \underline{\mathbb{Z}}\right) \Longrightarrow \pi_{t-s-V} MU^{(C_8)}.$$

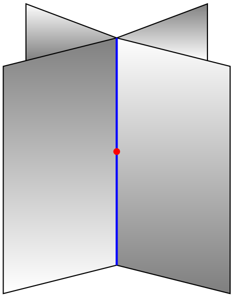
Computing with Slices

Key Fact

The E_2-term can be computed from equivariant simple chain complexes.

Cellular Chains for $S^{\rho_4 - 1}$

$$\overbrace{\underbrace{\mathbb{Z} \leftarrow \mathbb{Z}^2}_{C_\bullet(S^1)} \leftarrow \mathbb{Z}^4 \leftarrow \mathbb{Z}^4}^{C_\bullet(S^3)}$$



Gaps

Theorem

For any non-trivial subgroup H of C_8 and for any induced sphere $Ind_K^{C_8} S^{k\rho_K}$ with $k \in \mathbb{Z}$,

$$H^K_{-2}(S^{k\rho_K}; \underline{\mathbb{Z}}) = 0$$

1. If $k \geq 0$ or $k < -2$, then $C_{-2} = 0$
2. If $k = -1, -2$, in the relevant degrees, the complex is $\mathbb{Z} \to \mathbb{Z}^2$ by $1 \mapsto (1, 1)$.

Corollary

For any k, $\pi_{-2}\Sigma^{k\rho_8} MU^{(C_8)} = 0$.

Homotopy Fixed Points & Periodicity

Euler and Orientation Classes

Homology of representation spheres is generated by Euler classes and orientation classes for representations.

Theorem

1. *The fixed and homotopy fixed points of $\Omega_{\mathbb{O}}$ agree if all Euler classes are nilpotent.*
2. *The homotopy of the homotopy fixed points of $\Omega_{\mathbb{O}}$ is periodic if some regular representation is orientable.*