Computations towards $K(ko)$

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Outline

Basic Setup

- $K(-)$ is a functor from commutative rings / commutative S-algebras to commutative S-algebras.
- $K(-)$ is *very* hard to compute but has wide-ranging applications.
- Basic step: Approximate $K(A)$ by $THH(A) = Tor^{A \otimes A^{op}}(A, A)$.

From *THH* to *K*

- $THH(A)$ has a natural action of S^1
- There is a natural map $K(A) \to THH(A)$ that factors through the S^1 fixed points.
- At p, the S^1 fixed points are essentially $TC(A)$.
- For connective A, $K(A) \to TC(A)$ is almost an equivalence.

Chromatic Red Shift

Conjecture

If A is a ring spectrum of type n, then $K(A)$ has type $n+1$.

Examples of type n ring spectra:

- $H\mathbb{Q}$ (n=0)
- ℓ (n=1)
- $BP\langle n\rangle$.

$V(0)$ and $V(1)$ Homology of $THH(\ell)$

- Bökstedt / McClure-Staffeldt: Computed $H_* THH(\ell)$, effectively also giving the $V(1)$-homology:

$$V(1)_* THH(\ell) = E(\lambda_1, \lambda_2) \otimes \mathbb{F}_p[\mu_2].$$

- McClure & Staffeldt: Computed the $V(0)$-homology: Everything but λ_1 is v_1-torsion.
- Angeltveit & Rognes: Extended these results to all primes.

The $V(1)$ Homology of $K(\ell)$

- Ausoni & Rognes: Computed the $V(1)$ homotopy of $K(\ell)$ for $p > 3$ in several steps:
  - Calculate $V(1)_* THH(\ell)^{C_{p^n}}$ and the Tate analogue.
  - Use this to calculate $V(1)_* TC(\ell)$.
  - Used the trace map $V(1)_* K(\ell) \to V(1)_* TC(\ell)$ to understand $V(1)_* K(\ell)$.
- Extended by Ausoni to $p = 3$ and to ku.

$THH_*(\ell)$ and $THH_*(ko)$

Theorem

p-locally, $\overline{THH}_*(\ell) = F \oplus T$

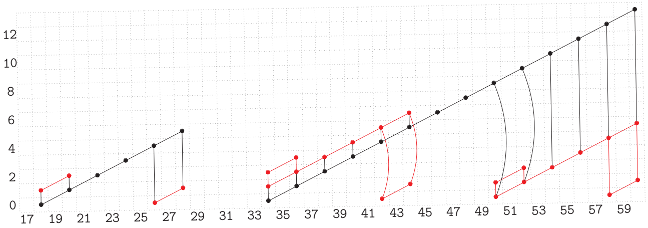
- F is p-torsion free and v_1 becomes increasingly p divisible.
- T is p and v_1-torsion and is a sum of self-dual modules.

Theorem

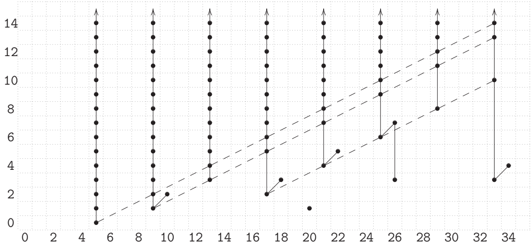
$$0 \to F_{ko} \to \overline{THH}_*(ko) \to T_{ko} \to 0$$

- F_{ko} is η^2-torsion and v_1^4 becomes increasingly 2 divisible.
- T_{ko} is 2, η, and v_1^4 torsion and is a sum of self-dual modules.

A Picture of 2-torsion in $THH_*(ku)$



A Picture of $THH_*(ko)$



Proving These Results

- $THH(-)$ has a relative version $THH(A; M)$.
- ℓ has 4 natural bimodules: ℓ, ℓ/p, $H\mathbb{Z}_{(p)}$, and $H\mathbb{F}_p$.
- These give Bockstein spectral sequences for adding in p or v_1.
- The end results must be the same.
- *ko* has one extra bimodule: *ku*.

Lifting Beyond $V(1)$

- There is an associative ring spectrum Y_2 such that

$$H_*(Y_2) = \mathbb{F}_p[\xi_1, \xi_2] \otimes E(\tau_0, \tau_1).$$

- ${Y_2}_*(THH(\ell)) = \mathbb{F}_p[\xi_1, \xi_2, \mu_2] \otimes E(\lambda_1, \lambda_2)$
- For big enough values of p, Y_2 is filtered by $V(1) \Rightarrow$ Can play a similar game to understand homotopy and Tate fixed points.

Future Work

- Replace "magical" and subtle theorem used by Rognes and Ausoni to understand all homotopy fixed points.
- Compute $Y_{2*}(THH(\ell)^{hS^1})$ and $Y_{2*}TC(\ell)$.
- Run the Y_2-based Adams spectral sequence to get $TC_*(\ell)$.

Summary

- Can understand $THH(\ell)$ and $THH(ko)$.
- Can use a special spectra Y_2 to try to compute $TC(-)$.