THE HOMOTOPY OF $EO_{2(p-1)}$

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1. Introduction

In this note, we will outline the arguments used to compute the homotopy of the spectrum $EO_{f(p-1)}(\mathbb{Z}/p)$, providing an essentially complete argument in the case $f = 2$. Many of the key features of the full argument are present here: the use of infinite descent and the use of equivariant methods to produce geometric differentials. Our primary purpose is to present the basic ideas while we finish writing the full forms.

2. Group Action

We have the following description of the cohomology of $\mathbb{Z}/p$ with coefficients in the Lubin-Tate ring.

Theorem 1 ([3])**.** *Modulo the image of the transfer map,*

$$H^*(\mathbb{Z}/p; E_{2(p-1)*}) = \mathbb{F}_{p^n}[\![\delta_1]\!][\beta][\Delta^{\pm 1}] \otimes E(h_{1,0}, h_{2,0}),$$

together with a class on the zero line of the form $h_{1,0}h_{2,0}/\beta$, where β is the periodicity generator for $H^2(\mathbb{Z}/p;\mathbb{Z})$ and has bidegree $(-2, 2)$, Δ^{-1} is the multiplicative norm of a distinguished invertible element in degree -2 (essentially u), δ_1 is a multiplicative norm in degree 0, and $h_{i,0}$ are exterior classes with the usual May names.

The argument in [3] is stronger than we need in this context. We can use Ravenel's "Method of Infinite Descent" to deduce the group cohomology and the names of the elements [6, Chapter 7]. We summarize these results here.

The formulas of Devinatz and Hopkins [2] on the action of the Morava stabilizer group on the Lubin-Tate ring allow us to show that $\mathbb{Z}/p$-equivariantly, $E_{f(p-1)*}T(f)$ has a very simple form. Let

$$R = \mathbb{W}(\mathbb{F}_{p^n})[t_1, \sigma(t_1), \ldots, \sigma^{p-1}(t_1), t_2, \ldots, \sigma^{p-1}(t_f)][\bar{\Delta}^{-\frac{1}{p^f-1}}],$$

where $\bar{\Delta} = N(\sigma(t_f) - t_f) \cdots N(\sigma^{p-1}(t_f) - t_f)$, and let $S = \mathbb{F}_{p^n}[s_1, \ldots, s_f]$ (viewed as an ungraded ring). The ring S becomes a $\mathbb{Z}/p$-algebra by declaring s_i to be invariant for $i < f$, and $\sigma(t_f) = t_f + 1$. We also get a canonical equivariant map $R \to S$ which sends t_i to s_i, and let I denote the kernel of this map. Devinatz and Hopkins' formulas show three things:

(1) S is $\mathbb{Z}/p$-equivariantly isomorphic to $E_{f(p-1)*}T(f)/\mathfrak{m}$.
(2) $R_I^\wedge$ is $\mathbb{Z}/p$-equivariantly isomorphic to $E_{f(p-1)*}T(f)$.
(3) $R_I^\wedge \to S$ is reduction modulo $\mathfrak{m}$.

We pause here to remark about the differences between this and the group action theorem. This result hinges on finding a copies of the regular representation. If we reduce modulo p and the square of the maximal ideal $\mathfrak{m}$ of $E_{f(p-1)*}$, then we see that t_i generates a copy of the regular representation for all i. In particular, since regular representations are rigid, we conclude R is essentially $E_{f(p-1)*}T(f)$.

We can now use Ravenel's method of descent to convert this information into group cohomology information for $EO_{f(p-1)*}$ [6]. The underlying idea is to pretend that $T(i)$ is an E_∞ ring spectrum and to resolve a $T(i)$-module by $T(i+1)$ in the category of $T(i)$-modules. More specifically, the idea is to form the usual cosimplicial Adams resolution of Baker and Lazarev [1]:

$$M \wedge_{T(i)} T(i+1) \rightrightarrows M \wedge_{T(i)} T(i+1) \wedge_{T(i)} T(i+1) \dots .$$

In our case, the E_2 term for computing the $\mathbb{Z}/p$-cohomology of $E_{2(p-1)*}T(1)$ (modulo the image of the transfer map) is given by

$$\mathbb{Z}_{p^n}[N(t_1)][\beta][\Delta^{\pm 1}] \otimes E(h_{2,0}),$$

since $N(t_2)^{p^{i-1}}$ annihilated $h_{2,i}$. There is no room for any differentials, so we conclude that this is E_∞.

Repeating this for $T(0) = S^0$ homology, we see that the E_2 term for computing the $\mathbb{Z}/p$-cohomology of $E_{2(p-1)*}$ is given by

$$\mathbb{Z}_{p^n}[N(\sigma t_1 - t_1)][\beta][\Delta^{\pm 1}] \otimes E(h_{1,0}, h_{2,0}).$$

For degree reasons, we again conclude that all of the classes present here are permanent cycles, and therefore $E_2 = E_\infty$. This shows that the cohomology of $\mathbb{Z}/p$ with coefficients in $E_{2(p-1)*}$ is the action described above. The class $N(\sigma t_1 - t_1)$ is a Δ translate of the class δ_1 described above.

3. Differentials

3.1. **The differentials on β, on Δ, and on $h_{2,0}$.** The key source of differentials in homotopy fixed point spectral sequences is geometric: find some equivariant map from a finite spectrum X into E_n, pass to homotopy fixed points, and analyze the attaching maps of the cells in X^{hG}. This cellular filtration coincides with the homotopy fixed point filtration (for X and G sufficiently simple), so we get a map of spectral sequences which we can readily understand.

As an easy, immediate example, the unit map $S^0 \to E_n$ is an equivariant map (for any subgroup of the Morava stabilizer group). If we take $G = \mathbb{Z}/p$, then this map induces a map of homotopy fixed points ${S^0}^{h\mathbb{Z}/p} \to EO_n$. Since S^0 has a trivial $\mathbb{Z}/p$ action, the source of this map is the Spanier-Whitehead dual of $B\mathbb{Z}/p_+$, and the top cell, which is unattached, goes in via 1.

In the map of homotopy fixed point spectral sequences, the fundamental class of the $(-2k)$-cell hits $\beta^k \cdot 1$ (and this is essentially generic). Now we use the cellular structure of the source to conclude our first differential. Milnor showed that the class $b \in H^2(B\mathbb{Z}/p)$ has completed $\mathcal{A}_*$-coaction given by

$$b \mapsto \sum_{i=0}^{\infty} \xi_i \otimes b^{p^i},$$

where ξ_i is the usual Milnor generator of $\mathcal{A}_*$ [4]. In particular, we see that b^p and b are linked by a ξ_1-coaction. The class $\xi_i^{p^j}$ detects by $h_{i,j}$, by definition, so we conclude that there is a differential in the homotopy fixed point spectral sequence for S^0 of the form

$$d_{2p-1}(b) = h_{1,0}b^p.$$

The equivariant unit map then produces for us our first homotopy fixed point differential.

Theorem 2. *There is a d_{2p-1}-differential on β:*

$$d_{2p-1}(\beta) = h_{1,0}\beta^p.$$

This geometric argument shows much more, actually. We see that that there is a kind of "total differential" on β of the form

$$d_*(\beta) = \sum_{i=1}^{\infty} h_{i,0}\beta^{p^i}.$$

The differential visible in any homotopy fixed point spectral sequence built out of EO_n will then see the first non-vanishing term of this, and the target is necessarily a permanent cycle. In fact, this argument is a universal one: for any E_∞ ring spectrum on which $\mathbb{Z}/p$ acts via E_∞ self-maps, there is a differential on the periodicity generator of this form.

We can apply a very similar argument to get differentials on classes which are multiplicative norms and on classes which are in H^1. Let N be the multiplicative norm of a class u of dimension $2k$. The class u represents a map $S^{2k} \to E_n$, so N represents a $\mathbb{Z}/p$-equivariant map

$$S^{k\rho} \to E_n,$$

where ρ is the complex regular representation of $\mathbb{Z}/p$. Passing to homotopy fixed points produces yields a map of filtered spectra

$$S^{k\rho^{h\mathbb{Z}/p}} \to EO_n.$$

The homotopy fixed point spectrum of $S^{k\rho}$ is the Spanier-Whitehead dual of the Thom spectrum of $-k\rho \times_{\mathbb{Z}/p} E\mathbb{Z}/p$ over $B\mathbb{Z}/p$. To understand the differentials on Δ, we have to understand how the top cell of this dual Thom spectrum attaches to lower cells. We do so by looking at the Thom class.

Since we are working over $\mathbb{C}$, the representation ρ splits into a sum of 1-dimensional representations:

$$\rho = \underline{1} \oplus \lambda \oplus \lambda^2 \oplus \cdots \oplus \lambda^{p-1},$$

where λ is the inclusion $\mathbb{Z}/p = \mu_p \subset \mathbb{C}^*$. When we form the associated vector bundle, we conclude that

$$k\rho \times_{\mathbb{Z}/p} E\mathbb{Z}/p = \bigoplus^{k} \left(\underline{1} \oplus L \oplus \cdots \oplus L^{p-1}\right),$$

where L is the canonical line bundle on $B\mathbb{Z}/p$. We can also determine how the top cell in this dual Thom spectrum by considering a trick similar to Milnor's. In the Thom spectrum for the canonical bundle, we have that the coaction on the Thom class is given by

$$1 \otimes m + \sum \xi_i \otimes b^{p^i-1}m.$$

The Thom class is multiplicative, and L^j corresponds to pulling back along the multiplication by j map. Since this sends b to jb, and since $j^{p^i-1} = 1$ working modulo p, we conclude that if m is the Thom class of $k\rho$, then the completed $\mathcal{A}_*$-coaction on m is given by

$$\left(1 \otimes 1 + \sum \xi_i \otimes b^{p^i-1}\right)^{-k(p-1)} m.$$

Just as with β, this gives us the differentials on Δ.

Theorem 3. *There is a d_{2p-1} differential on Δ of the form*

$$d_{2p-1}(\Delta) = h_{1,0}\beta^{p-1}\Delta.$$

Proof. Let b_i denote the fundamental class of the $(-2i)$-cell in the homotopy fixed point spectral sequence for $S^{k\rho}$. Under the map of homotopy fixed point spectral sequences, $b_i \mapsto \beta^i\Delta$. In completed $\mathcal{A}_*$-coaction shows us that the Thom class supports a $\mathcal{P}^1$, and this shows that in the homotopy fixed point spectral sequence for $S^{k\rho}$, we have a differential

$$d(b_0) = h_{1,0}b_{p-1}.$$

Naturality then implies that we have the desired differential on Δ:

$$d_{2p-1}(\Delta) = h_{1,0}\beta^{p-1}\Delta. \qquad \square$$

We remark that just as with β, there is an underlying total differential involving terms of the form $h_{i,0}\beta^{p^i-1}\Delta$.

The final case is to understand the differentials on $h_{2,0}$. The argument is almost exactly the same as for Δ and β. The main difference is the complex we map in. Let $S^k \to S^k \wedge S^0[\mathbb{Z}/p_+] = S^k \wedge \mathbb{Z}/p_+$ denote the transfer map, and let $S^k[\bar{\rho}]$ denote the equivariant cofiber.

If we have an element $u \in \pi_k E$, then we get a natural equivariant map

$$S^k \wedge S^0[\mathbb{Z}/p_+] \to E.$$

If $Tr(u) = 0$, then we know that this equivariant map factors through $S^k[\bar{\rho}]$. We therefore get a map of homotopy fixed point spectra

$$S^k[\bar{\rho}]^{h\mathbb{Z}/p} \to E^{h\mathbb{Z}/p},$$

and an associated map of homotopy fixed point spectral sequences. At this point, the argument is exactly like the argument for β or for Δ.

Theorem 4. *There is a d_{2p-1}-differential on $h_{2,0}$ of the form*

$$d_{2p-1}(h_{2,0}) = h_{1,0}\beta^{p-1}h_{2,0}.$$

We close this subsection with a remark about Δ translates of β. Any power of Δ is a norm, so we can apply the same argument. In particular, we can analyze the "total" differential on Δ and on any of its β multiples. Degree considerations then show us the following.

Proposition 1. *The class $\beta\Delta^{1/(p-1)}$ is a permanent cycle.*

Since Δ has no roots, some explanation is required. This expression simply means that if Δ^{p^k} is a permanent cycle, then $\beta\Delta^{-1-p-\cdots-p^{k-1}}$ is also. Let b denote the corresponding permanent cycle. In the course of the computation, we will see exactly what power of Δ is needed.

3.2. **The differential on δ_1.** We quickly show that δ_1 (and in fact, all classes which are norms of degree zero classes) is a permanent cycle. In fact, the argument is essentially the same as showing that the unit map is a permanent cycle.

Theorem 5. *If Nu is a multiplicative norm in degree 0, then Nu is a non-bounding permanent cycle in the homotopy fixed point spectral sequence.*

Proof. The norm argument produces an equivariant map of the form

$$S^{0\rho} = S^0 \to E.$$

Passing to homotopy fixed points yields a map

$$D(B\mathbb{Z}/p_+) = (S^0)^{h\mathbb{Z}/p} \to E^{h\mathbb{Z}/p},$$

and the norm map is carried by the 0-cell. Since this is the disjoint base point, we conclude that it is a permanent cycle in the homotopy fixed point spectral sequence for $(S^0)^{h\mathbb{Z}/p}$, and therefore a permanent cycle in the homotopy fixed point for EO_{p-1}. □

In particular, we conclude that any element like δ_1 (so any element of the form $N(u_i)$ in the higher homotopy fixed point spectral sequences) is a permanent cycle.

3.3. **The differential on Δ^p.** The previous discussion of the geometric differentials shows that the leading term of the differential on Δ^p has the form

$$d(\Delta^p) = h_{1,1}\beta^{p^2-p}\Delta^p.$$

However, $h_{1,1}$ is not present on the 1-line. We can identify this with elements of higher filtration, using a cohomological version of the $EO_{2(p-1)}$ based Atiyah-Hirzebruch spectral sequence for $T(1)$. The spectrum $T(1)$ has the property that $BP_*T(1) = BP_*[t_1]$, and the class $t_1^{p^i}$ is attached to the bottom cell by $h_{1,i}$. There is therefore an algebraic Atiyah-Hirzebruch spectral sequence of the form

$$E_1 = H^*(\mathbb{Z}/p; E_{2(p-1)*})[t_1] \Longrightarrow H^*(\mathbb{Z}/p; E_{2(p-1)*}T(1)).$$

The d_1-differential on t_1 is given by $d_1(t_1) = h_{1,0}$, since t_1 detects $h_{1,0}$. For degree reasons, the algebraic spectral sequence then collapses, giving us the homotopy fixed point spectral sequence E_2 term for the homotopy of $EO_{2(p-1)} \wedge T(1)$ (modulo the image of the transfer):

$$E_2 = E(h_{2,0}) \otimes \mathbb{F}_{p^n}[\beta, \Delta^{\pm 1}, t_1^p].$$

Ravenel has shown that $h_{2,0}b_{2,0}^p = 0$ in the homotopy of $T(1)$ [5]. This means that we must have an identical relation in the $EO_{2(p-1)}$-homology of $T(1)$. From the above discussion, we know that

$$d_{2p-1}(\Delta) = h_{1,0}\beta^{p-1}\Delta = 0,$$

and Δ therefore survives through the E_{2p}-page. For degree reasons, we therefore must have a differential on t_1^p.

Theorem 6. *There is a d_{2p-1}-differential on t_1^p of the form*

$$d_{2p-1}(t_1^p) = h_{2,0}\beta^{p-1}.$$

This differential is morally similar to the differentials on Δ, β, and $h_{2,0}$. Part of the proof of the general theorem involves a generalization of this: in the homotopy fixed point spectral sequence for $T(i) \wedge EO_{f(p-1)}$, there is a differential of the form

$$d_{2p-1}(t_i^p) = h_{i+1,0}\beta^{p-1}.$$

This is closely tied to the failure of $T(i)$ to be an E_∞-ring spectrum.

There is a corresponding Toda-style differential:

$$d(h_{2,0}t_1^{p(p-1)}) = b_{2,0}\beta^{(p-1)^2}.$$

Since $b_{2,0}$ is a Δ-translate of β, this produces a horizontal vanishing line in the spectral sequence and the spectral sequence collapses.

Corollary 1. *In the homotopy fixed point spectral sequence for $EO_{2(p-1)}$,*

$$h_{1,1} = h_{2,0}\beta^{p-1}.$$

Corollary 2. *There is a d_{2p^2-1}-differential on Δ^p:*

$$d_{2p^2-1}(\Delta^p) = h_{2,0}\beta^{p^2-1}\Delta^p.$$

This allows us to fully compute the homotopy of $EO_{2(p-1)}$.

4. Computation of the Homotopy

Since all of the algebraic generators with the exception of δ_1 support differentials, we will write everything in terms of permanent cycles we will deduce from the differentials on Δ and Δ^p.

Since the differentials on Δ and $h_{2,0}$ have the same sign, the Leibnitz rule shows that the class $a_2 = h_{2,0}\Delta^{-1}$ is a cycle through at least d_{2p-1}. We also already saw that $b = \beta\Delta^{1+p}$ and δ_1 are cycles (though b only through the range Δ^{p^2} is). This lets us rewrite the E_2-term (as always, modulo the image of the transfer) as

$$E(h_{1,0}, a_2) \otimes \mathbb{F}_p[b] \otimes \mathbb{F}_p[\![\delta_1]\!][\Delta^{\pm 1}].$$

The d_{2p-1} differential is

$$d_{2p-1}(\Delta) = h_{1,0}\beta^{p-1}\Delta = h_{1,0}b^{p-1}\Delta^{p^2}.$$

Every term on the right-hand side is a permanent cycle, and we can apply a Toda style argument to see that

$$d_{2(p-1)^2+1}(h_{1,0}\Delta^{p-1}) = b_{1,0}b^{(p-1)^2}\Delta^{p^2(p-1)},$$

where

$$b_{1,0} = \langle \underbrace{h_{1,0}, \ldots, h_{1,0}}_{p} \rangle = \delta_1\beta\Delta^{p-1}.$$

This last expression reduces to $\delta_1 b\Delta^{2p}$, so we conclude that the differential is

$$d_{2(p-1)^2+1}(h_{1,0}\Delta^{p-1}) = \delta_1 b^{(p-1)^2+1}\Delta^{p^2(p-1)+2p}.$$

This leaves a large collection of cycles: δ_1, Δ^p, b, a_2, and $h_{1,0}\Delta^i$ for $0 \leq i \leq p-2$.

We now consider the differential on Δ^p:

$$d_{2p^2-1}(\Delta^p) = h_{2,0}\beta^{p^2-1}\Delta^p = a_2 b^{p^2-1}\Delta^{p^3+p^2}.$$

All of the terms with the exception of a_2 are visibly cycles, so we conclude that a_2 must be as well (the b-torsion at this stage was controlled by differentials for which

a_2 was known to be a cycle). We could actually conclude this from an argument similar to that of $\beta\Delta^{1/(p-1)}$ being a permanent cycle: were such a thing to make sense, $h_{2,0}\beta^{-1}$ is a permanent cycle. Thus

$$h_{2,0}\beta^{p-1} = h_{2,0}b^{p-1}\Delta^{p^2-1} = a_2 b^{p-1}\Delta^{p^2}$$

is a cycle for as long as β^p is (in fact, we get much more, since the differential on β^p is essentially that of Δ^p and the target is $h_{2,0}$-torsion). In particular, we see that there could not be any lower differentials.

Since $b^{p^2-1}h_{1,0} = 0$, we conclude that all of the Δ^p multiples of the classes $h_{1,0}\Delta^i$ survive. Since $\delta_1 b^{p^2-1} = 0$, we also conclude that the Δ^p multiples of δ_1 also survive.

There is a Toda style differential originating from this differential:

$$d_{2(p^2-1)(p-1)+1}(a_2\Delta^{p(p-1)}) = b_2 b^{(p^2-1)(p-1)}\Delta^{p^2(p^2-1)},$$

where again

$$b_2 = \langle\underbrace{a_2, \dots, a_2}_{p}\rangle = \Delta^{-p}\beta\Delta^{p^2-1} = b\Delta^{p^2}.$$

This shows that the higher differential has the form

$$d_{2(p^2-1)(p-1)+1}(a_2\Delta^{p(p-1)}) = b^{1+(p^2-1)(p-1)}\Delta^{p^4}.$$

In particular, we conclude that the spectral sequence has a horizontal vanishing line at $s = 2 + 2(p-1)(p^2-1)$, and $E_s = E_\infty$.

References

1. Andrew Baker and Andrej Lazarev, *On the Adams spectral sequence for R-modules*, Algebr. Geom. Topol. **1** (2001), 173–199 (electronic). MR MR1823498 (2002a:55011)
2. Ethan S. Devinatz and Michael J. Hopkins, *The action of the Morava stabilizer group on the Lubin-Tate moduli space of lifts*, Amer. J. Math. **117** (1995), no. 3, 669–710.
3. Michael A. Hill, Michael J. Hopkins, and Douglas C. Ravenel, *The action of finite subgroups of the Morava stabilizer group on the Lubin-Tate space of lifts*, In progress.
4. John Milnor, *The Steenrod algebra and its dual*, Ann. of Math. (2) **67** (1958), 150–171. MR MR0099653 (20 #6092)
5. Douglas C. Ravenel, *The first Adams-Novikov differential for the spectrum $T(m)$*, Available on the author's website.
6. ———, *Complex cobordism and stable homotopy groups of spheres*, Pure and Applied Mathematics, vol. 121, Academic Press Inc., Orlando, FL, 1986.