QUADRATIC FORMS BEYOND ARITHMETIC

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1. Introduction

The concept of quadratic forms can be traced back to ancient civilizations such as the Babylonians and Greeks. The Greeks, particularly Euclid in his famous work "Elements," presented geometric methods for solving quadratic equations. The Greeks' focus on geometry and their methods continued to influence mathematicians for centuries. Rules for quadratic equations were also discussed in 'The Nine Chapters on the Mathematical Art' composed in China by 200 BCE. The general formula for solving a quadratic equation in one variable - equivalent to the modern symbolic formula - was first stated by the Indian mathematician Brahmagupta in his treatise Brahmasphutasiddhanta in 628 CE. [1]

A quadratic form over a commutative ring R is a homogeneous polynomial $\sum a_{ij}x_ix_j$ of degree 2 in n variables $x_1, \ldots, x_n$ with coefficients a_{ij} in R. In particular, the sum of squares $x_1^2 + x_2^2 + \ldots + x_n^2$ is a quadratic form defined over any R.

The problem of representing integers as sums of squares dates back to ancient times. The Greeks, especially the Pythagoreans, were interested in the properties of numbers and their geometric interpretations. The concept of sums of squares is closely related to the Pythagorean theorem, conceived in Mesopotamia (1800 BC), first stated precisely in the Shulbha Sutra of Baudhayana (800 BC) and a statement of proof from China.[2]

In the 7th century, the Indian mathematician Brahmagupta considered what is now called Pell's equation, $x^2 - ay^2 = 1$, and found a method for its solution.

One of the earliest and most significant results in the area of quadratic forms is Fermat's theorem on sums of two squares. In the 17th century, Fermat stated that an odd prime number p can be expressed as a sum of two squares if and only if p is congruent to 1 modulo 4.

Another milestone in the study of sums of squares is the Four Square Theorem, proven by Lagrange in 1770. This theorem states that every positive integer can be represented as the sum of four squares.

Euler's Sums of Two Squares Identity

$$(x_1^2 + x_2^2)(y_1^2 + y_2^2) = (x_1y_1 - x_2y_2)^2 + (x_1y_2 + x_2y_1)^2$$

[1]https://en.m.wikipedia.org/wiki/Brahmagupta

[2]https://m.rediff.com/news/special/did-india-discover-pythogoras-theorem-a-top-mathematician-answers/20150109.htm

shows that the set of sums of two squares in a commutative ring is closed under multiplication; this statement was generalized to the binary quadratic form x^2+ny^2 by Brahmagupta. Similar formulas also exist for the sums of 4 and 8 squares.

In the beginning of the 19th century Gauss completed the theory of composition of binary quadratic forms over the integers. Far-reaching generalizations in the 21st century are the higher composition laws of Bhargava leading to asymptotics for the number of number fields with bounded discriminant of degree at most 5 (see [1]).

In the early 20th century, the main focus of the study of quadratic forms is the arithmetic theory over the rings of algebraic integers and number fields. Witt in the 30's laid the foundation of the *algebraic theory* of quadratic forms that deals with the study of quadratic forms over *arbitrary* fields. There was great progress in this study initiated by Pfister's theory of multiplicative forms in the 60's. Several open questions and conjectures were posed since then, a major one being the Milnor Conjectures. Since the 80's, the introduction of powerful techniques from algebraic geometry transformed the study of quadratic forms, leading to great leaps like a solution of the Milnor Conjecture.

In this article we trace the progress in the algebraic theory of quadratic forms over the last four decades. For simplicity we only consider fields of characteristic different from 2, although the theory of quadratic forms in characteristic 2 is also well developed. We stay aside from the theory of quadratic forms over commutative rings such as rings of algebraic integers etc.

In Section 2 we briefly recall basic definitions. In the next section we introduce a method of the study of quadratic forms in the last four decades based on algebraic geometry. We introduce the quadric hypersurface (quadric) X_q associated with a quadratic form q and a tool based on the study of closed subvarieties (algebraic cycles) on the products of quadrics. In Section 4 we state two theorems on the proof of Milnor Conjectures that compare the graded Witt ring and the graded mod 2 Galois cohomology via Milnor's K-theory of fields.

In the next section some discrete invariant of fields (the u-invariant and the Pythagoras number) that are defined by means of quadratic forms are considered. Although these invariant were defined a while ago, the newly developed techniques allowed us to obtain recent results in this direction. There are big open questions concerning these invariants. The determination of the u-invariant of function fields of curves over totally imaginary number fields is the ultimate goal while even the finiteness for $\mathbb{Q}(\sqrt{-1})$ remains open. Another big open question is the determination of the Pythagoras number of rational function fields over the field of real numbers and the question is open even for $\mathbb{R}(x_1, x_2, x_3)$.

Another discrete invariant, the dimension of a quadratic form, is discussed in Section 6. One of the old intriguing questions on the determination of all possible splitting patterns of quadratic forms of given dimension is still unsolved.

In the last section we introduce the Chow Motives technique. Indecomposable direct summands of the motives of quadrics can be visualized in the diagrams defined in this section.

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2. Definitions

Basic references are [13] and [2].

Let F be a field of characteristic not 2, that is $1+1 \neq 0$ in F. We write $F^\times$ for the multiplicative group of nonzero elements in F.

A *quadratic form* on a finite dimensional vector space V over F is a map $q : V \to F$ such that
1) $q(xv) = x^2 q(v)$ for all $x \in F$ and $v \in V$ and
2) the map $b_q : V \times V \to F$ defined by the formula $b_q(v, v') = \frac{1}{2}[q(v+v') - q(v) - q(v')]$ is a (symmetric) bilinear form.

Note that the bilinear form b_q reconstructs q by the equality $q(v) = b_q(v, v)$. Thus, to give a quadratic form on a vector space V is the same as to give a symmetric bilinear form on V.

The integer $n = \dim(V)$ is called the *dimension* of the form q. Let $\{v_1, v_2, \dots, v_n\}$ be a basis for V. If $v = \sum x_i v_i$ with $x_i \in F$ is an arbitrary vector in V, we have

$$q(v) = \sum_{i,j} a_{ij} x_i x_j,$$

where $a_{ij} = a_{ji} = b_q(v_i, v_j)$. Thus q is given by a quadratic homogeneous polynomial over F. We say that q is *nondegenerate* if the (symmetric) $n \times n$ matrix $A = (a_{ij})$ is nondegenerate. This is equivalent to the nondegeneracy of the bilinear form b_q. Below we will consider nondegenerate quadratic forms only.

The determinant of A is well defined modulo the subgroup $F^{\times 2}$ of squares in $F^\times$. The *discriminant* of q is $(-1)^{\frac{n(n-1)}{2}} \det(A)$ in $F^\times / F^{\times 2}$.

Two quadratic forms $q : V \to F$ and $q' : V' \to F$ are *isomorphic* if there is an F-linear isomorphism $f : V \to V'$ such that $q(v) = q'(f(v))$ for all $v \in V$. The forms q and q' are isomorphic if and only if they are given by the same quadratic polynomial in some bases for V and V'.

Let $a_1, a_2, \dots, a_n$ be nonzero elements in F. We write $\langle a_1, a_2, \dots, a_n \rangle$ for the quadratic form $\sum a_i x_i^2$ on the space of n-columns $V = F^n$. It is called the *diagonal form* and it has a diagonal matrix in the standard basis for F^n. Every quadratic form over F is isomorphic to a diagonal form $\langle a_1, a_2, \dots, a_n \rangle$ for some (not uniquely determined) $a_i \in F^\times$.

If $q : V \to F$ is a quadratic from over F and L/F is a field extension, one can define a quadratic form q_L on the L-space $L \otimes_F V$ over L with the associated bilinear form b on $L \otimes_F V$ defined by $b(x \otimes v, x' \otimes v') = xx' b_q(v, v')$ for all $x, x' \in L$ and $v, v' \in V$.

The *orthogonal sum* of two forms $q : V \to F$ and $q' : V' \to F$ is the quadratic form $q \perp q'$ on $V \oplus V'$ defined by $(q \perp q')(v, v') = q(v) + q'(v')$.

The form $\mathbb{H} = \langle 1, -1 \rangle$ is the *hyperbolic plane*. A quadratic form is *hyperbolic* if it is isomorphic to the orthogonal sum $n\mathbb{H}$ of $n > 0$ copies of $\mathbb{H}$.

A quadratic form $q : V \to F$ is called *isotropic* if there is a nonzero vector $v \in V$ such that $q(v) = 0$; otherwise, q is called *anisotropic*. Every form q is isomorphic to $q_{an} \perp k\mathbb{H}$, where q_{an} is an anisotropic form and $k \geqslant 0$. The integer $w(q) = k$ is called the *Witt index* of q. It is equal to 0 if and only if q is anisotropic.

Two quadratic forms q and q' are *Witt equivalent* if the forms q_{an} and q'_{an} are isomorphic. The set $W(F)$ of equivalence classes of quadratic forms over F is the *Witt ring* of F with respect to the orthogonal sum and tensor product of forms. For example, $W(\mathbb{R})$ is isomorphic to $\mathbb{Z}$: an integer $n > 0$ corresponds to the form $n\langle 1\rangle$ under the isomorphism.

One of the most important properties of the Witt ring: two quadratic forms q and q' are isomorphic if and only if $q = q'$ in $W(F)$ and $\dim(q) = \dim(q')$.

The ideal $I(F)$ of even dimensional forms in $W(F)$ is called the *fundamental ideal*. The quotient $W(F)/I(F)$ is isomorphic to $\mathbb{Z}/2\mathbb{Z}$. The powers $I^n(F)$ of the fundamental ideal form a filtration on $W(F)$. The discriminant yields an isomorphism between $I(F)/I^2(F)$ and $F^\times/F^{\times 2}$. The determination of all the quotients $I^n(F)/I^{n+1}(F)$ is one of the fundamental problems of the algebraic theory of quadratic forms (see Section 4 below).

Let $a_1, a_2, \dots, a_n \in F^\times$. The 2^n-dimensional quadratic form

$$\langle\langle a_1, a_2, \dots, a_n\rangle\rangle = \langle 1, -a_1\rangle \otimes \langle 1, -a_2\rangle \otimes \cdots \otimes \langle 1, -a_n\rangle$$

is called an *n-fold Pfister form*. The nth power $I^n(F)$ of the fundamental ideal in the Witt ring is generated by the n-fold Pfister forms as an abelian group. An n-fold Pfister form is either anisotropic or hyperbolic.

Example 2.1. 1) Let $K = F(a^{1/2})$ be a quadratic field extension of F. The *norm form* $q : K \to F$ defined by $q(x + ya^{1/2}) = x^2 - ay^2$ is the 1-fold Pfister form $\langle\langle a\rangle\rangle$.
2) Let $a, b \in F^\times$ and let Q be a *quaternion* 4-dimensional (associative noncommutative) F-algebra with basis $\{1, i, j, k\}$ and multiplication table $i^2 = a$, $j^2 = b$ and $k = ij = -ji$. The *reduced norm* form $q : Q \to F$ defined by $q(x + yi + zj + tk) = x^2 - ay^2 - bz^2 + abt^2$ is the 2-fold Pfister form $\langle\langle a, b\rangle\rangle$.
3) For a triple of elements $a, b, c \in F^\times$ there is an *octonian* (nonassociative noncommutative) 8-dimensional algebra C that admits a norm quadratic form $q : C \to F$ that is the 3-fold Pfister form $\langle\langle a, b, c\rangle\rangle$.

Note that norm forms q in all the examples are multiplicative, i.e., the product of two values of q is a value of q. By the Hurwitz Theorem, there are no algebras of dimensions not 1, 2, 4 and 8 with the multiplicative norm. Nevertheless, the set of nonzero values of an n-fold Pfister form is closed under multiplication for every n. In particular, the sum of 2^n squares is a multiplicative quadratic form over any field.

3. Algebro-geometric methods

During the last three decades the methods of algebraic geometry interfaced algebraic theory of quadratic forms. One can associate to every quadratic form $q : V \to F$ over F the quadric hypersurface X_q (simply called the *quadric* of q)

given by the equation $q(v) = 0$ in the projective space $\mathbb{P}(V)$. The quadric X_q has dimension $\dim(q) - 2$ and it is smooth if q is nondegenerate. The variety X_q is integral if $\dim(q) \geqslant 3$. By definition, X_q has a point over F if and only if the form q is isotropic. More generally, X_q has a point over a field extension L/F if and only if q is isotropic over L. For example, the quadric X_q, where $q = x^2 + y^2 + z^2$, has no points over $\mathbb{R}$, but it has points over $\mathbb{C}$.

The two quadrics X_q and X_h are isomorphic if and only if the quadratic forms q and h are *similar*, that is each of the forms q and h is an F-multiple of the other.

We write $F(q)$ for the function field $F(X_q)$. Note that q is isotropic over $F(q)$. Indeed, if $x_1, x_2, \dots, x_n$ are homogeneous coordinates in $\mathbb{P}(V)$, we can view the rational functions x_i/x_1 as the elements in $F(q)^\times$ satisfying $q(1, x_2/x_1, \dots, x_n/x_1) = 0$ in $F(q)$. Thus, the equation $q(v) = 0$ has a nonzero solution over the field $F(q)$.

The field $F(q)$ is a *generic splitting field* of q: the form q is isotropic over a field extension L/F if and only if there is an F-place from $F(q)$ to L, that is an F-algebra homomorphism of a valuation F-subalgebra $R \subset F(q)$ to L.

Consider the following example of an application of the theory of algebraic cycles. Let q be an anisotropic quadratic form over F and let L/F be a field extension. We would like to know when q is isotropic over L, i.e., the equation $q(v) = 0$ has a nonzero solution over L. Typically, the extension L/F is finitely generated, so we can choose an integral variety Y over F such that L is isomorphic to the function field $F(Y)$ of Y over F. By the main property of the quadric, X_q has a point over $L = F(Y)$ if and only if there is a morphism $\operatorname{Spec}(L) \to Y$ or, equivalently, a rational morphism $Y \dashrightarrow X_q$ defined on a nonempty open subset $U \subset Y$. The closure of the graph of $U \to X_q$ in the product $Y \times X_q$ yields a cycle class of dimension $d = \dim(Y)$ in the *Chow group* $\mathrm{CH}_d(Y \times X_q)$ of rational equivalence classes of cycles of dimension d on $Y \times X_q$.

Conversely, a prime algebraic cycle on $Y \times X_q$ of dimension d that is birational when projecting to Y yields a rational morphism $Y \dashrightarrow X_q$ and therefore a point of X_q over L.

Thus, one can use the machinery of algebraic cycles. In particular, one can obtain new cycles by intersecting with other cycles (for example, with the Chern classes of the vector bundles such as the tangent bundle), by considering pull-backs and push-forwards with respect to certain morphisms and also applying the Steenrod operations modulo 2.

4. Milnor Conjectures

The most spectacular achievement in the algebraic theory of quadratic forms is the solution of the Milnor Conjectures.

The *Milnor K-theory* $K_*^M(F)$ of a field F is the (graded) quotient ring of the tensor $\mathbb{Z}$-algebra of the multiplicative group $F^\times$ by the ideal generated by the tensors $a \otimes b$, where $a, b \in F^\times$ with $a + b = 1$. In particular, $K_0^M(F) = \mathbb{Z}$ and $K_1^M(F) = F^\times$. The group $K_n^M(F)$ is generated by the *symbols* $\{a_1, a_2, \dots, a_n\}$, where $a_i \in F^\times$, that are multiplicative with respect to every variable (when the other variables are fixed) and satisfy the *Steinberg relation* $\{a_1, a_2, \dots, a_n\} = 0$ if $a_i + a_j = 1$ for some $i \neq j$.

The assignment

$$\{a_1, a_2, \dots, a_n\} \mapsto \langle\langle a_1, a_2, \dots, a_n \rangle\rangle$$

yields a well defined graded ring homomorphism

$$s_F : K_*^M(F) \to I^*(F)/I^{*+1}(F).$$

Let Γ_F be the *absolute Galois group* of a field F, i.e., Γ_F is the Galois group of a separable closure of F over F. We write $H^*(F)$ for the graded cohomology ring of Γ_F with coefficients in $\mathbb{Z}/2\mathbb{Z}$. The multiplication in $H^*(F)$ is given by the cup-product $\cup$. In particular, $H^0(F) = \mathbb{Z}/2\mathbb{Z}$ and $H^1(F) = F^\times/F^{\times 2}$. For every $a \in F^\times$ write (a) for the corresponding cohomology class in $H^1(F)$. The assignment

$$\{a_1, a_2, \dots, a_n\} \mapsto (a_1) \cup (a_2) \cup \dots \cup (a_n)$$

yields a well defined graded ring homomorphism

$$h_F : K_*^M(F) \to H^*(F)$$

called the *norm residue homomorphism* modulo 2. Thus, we have the following diagram of graded ring homomorphisms

$$\begin{array}{ccc} & K_*^M(F)/2K_*^M(F) & \\ {}^{s_F}\swarrow & & \searrow^{h_F} \\ I^*(F)/I^{*+1}(F) & & H^*(F) \end{array}$$

Milnor has conjectured in the 70's that both maps s_F and h_F are isomorphisms for all fields. This conjecture for $n = 2$ was settled by the first author in the 80's leading to a solution of the longstanding question on the generation of the 2-torsion in the Brauer group of a field by quaternion algebras.

Voevodsky in [24] proved one of the Milnor Conjectures.

Theorem 4.1. *The graded ring homomorphism h_F is always an isomorphism.*

In the proof Voevodsky introduces a number of revolutionary ideas and tools. The main tool is the motivic cohomology defined by Voevodsky. Another tool is the motivic Steenrod operations defined by Voevodsky in analogy with the classical topological operations. The motivic analogs of some of the basic operations, the Milnor operations, played an essential role in the proof.

Enriching Voevodsky's methods, Orlov, Vishik and Voevodsky proved in [16] another Milnor Conjecture.

Theorem 4.2. *The graded ring homomorphism s_F is always an isomorphism.*

In particular, for every $n \geqslant 0$, we get a group isomorphism

$$c_F^n = h_F^n \circ (s_F^n)^{-1} : I^n(F)/I^{n+1}(F) \to H^n(F).$$

We can call the maps c_F^n the *cohomological invariants* of quadratic forms. These invariants determine anisotropic quadratic forms up to isomorphism as follows. Suppose we are given two anisotropic quadratic forms f and g and we want to decide whether f and g are isomorphic, or, equivalently, that the form $q = f \perp (-g)$ is hyperbolic. We compute the cohomological invariants c_F^n of q one by one starting with $n = 0$. Note that the next invariant is defined if the previous one vanishes.

The invariant c_F^0 takes q to its dimension modulo 2 in $\mathbb{Z}/2\mathbb{Z} = H^0(F)$. If this invariant vanishes, i.e., q has even dimension, then $q \in I(F)$. The invariant $c_F^1(q)$ is equal to the discriminant of q in $F^\times/F^{\times 2} = H^1(F)$. If the discriminant of q is trivial, the form q belongs to $I^2(F)$ and we can compute $c_F^2(q)$. This is the class of the Clifford algebra of q in the subgroup $\mathrm{Br}(F)[2] = H^2(F)$ of the Brauer group of classes of exponent 2. If this vanishes, then $q \in I^3(F)$ and so on. If $c_F^n(q) = 0$, i.e., $q \in I^{n+1}(F)$ and $\dim(q) < 2^{n+1}$, then q is hyperbolic ([19, p. 33]) and the form f and g are isomorphic.

5. Field Invariants

We discuss two integer invariants of fields associated to quadratic forms, namely the u-invariant and the Pythagoras numbers.

5.1. u-invariant.

The *u-invariant* $u(F)$ of a field F is the largest dimension of an anisotropic quadratic form over F. For example $u(\mathbb{R}) = \infty$ and $u(\mathbb{C}) = 1$. The u-invariant of a finite field is 2. The u-invariant of a local field is equal to 4. The Hasse-Minkowski Theorem implies that the u-invariant of a totally imaginary number field is 4. The formula $u(F((t))) = 2u(F)$ shows that every power of 2 is the u-invariant of some field. Breaking the myth that all invariants of fields associated to quadratic forms are a power of 2, it was shown in [15], that every even integer is the u-invariant of some field. On the other hand, it is known that the u-invariant is not equal to 3,5 or 7 ([19, Proposition 1.3, p. 111]). It is expected that every odd integer $\geqslant 9$ is the value of the u-invariant. Izhboldin has shown in [7] that there are fields of u-invariant 9. In [22] Vishik proved that every integer of the form $2^m + 1$ for $m \geqslant 3$ can be the u-invariant of a field.

The behavior of the u-invariant of finite field extensions has been studied extensively. We have the following theorem of Leep for finite field extensions.

Theorem 5.1. ([19, Theorem 3.1, p. 120]) Let L/F be an extension of degree n. Then $u(L) \leqslant \frac{n+1}{2}u(F)$.

There may be bounds for the u-invariants of finite extensions independent of the degree of the extensions.

The behavior of the u-invariant under rational function field extensions is not well understood.

Question 5.2. *If $u(F) < \infty$, is $u(F(t)) < \infty$?.*

If this question has an affirmative answer, then it follows that there are bounds independent of the degree for the u-invariant of finite field extensions. In fact the above question is wide open if F is a totally imaginary number field. Even for the function fields of p-adic curves, the question remained open until the late 90's when the first finiteness results emerged from Merkurjev and Hoffmann-Van Geel. A theorem of Saltman on bounding indices of elements in the 2-torsion of the Brauer group was pivotal to this theorem. We have the following theorem for function fields of p-adic curves.

Theorem 5.3. ([17], [18]) Let K be a p-adic field and F the function field of a curve over K. Then $u(F) = 8$.

We have the following more general extension of the above theorem which fundamentally uses a theorem of Heath-Brown on the zeros of systems of p-adic quadratic forms.

Theorem 5.4. ([14]) Let K be a p-adic field and F the function field of a variety of dimension d over K. Then $u(F) = 2^{d+2}$.

We also have the following theorem extending to general complete discretely valued fields, which uses the patching techniques of Harbater-Hartmann-Krashen.

Theorem 5.5. ([4]) Let K be a complete discretely valued field with residue field κ. Let F be the function field of a curve over K. Suppose that $\mathrm{char}(\kappa) \neq 2$ and there exists an integer d such that $u(L) \leqslant n$ for all finitely generated extensions of κ of transcendence degree at most 1. Then $u(F) \leqslant 2n$.

5.2. **Pythagoras number.** The *Pythagoras number* $p(F)$ of a field F is the smallest integer n such that every sum of squares in F is a sum of at most n squares in F.

For example $p(\mathbb{R}) = p(\mathbb{C}) = 1$. If F is a real number field, then $3 \leqslant p(F) \leqslant 4$ (cf. [19, p. 95]). The study of the Pythagoras number of fields which are formally real, i.e., -1 is not a sum of squares in the field, is interesting. The following theorem answers a question of Pfister.

Theorem 5.6. ([6]) Every positive integer $n \geqslant 1$ is the Pythagoras number of a formally real field.

The determination of Pythagoras numbers of rational function fields $\mathbb{R}(x_1, \cdots x_n)$ has a long history. Pfister theory leads to $p(\mathbb{R}(x_1, \cdots, x_n)) \leqslant 2^n$ (cf. [19, p. 95]). We have $p(\mathbb{R}(x)) = 2$ and $p(\mathbb{R}(x_1, x_2)) = 4$ ([19, p. 96]). We have bounds $n + 2 \leqslant p(\mathbb{R}(x_1, \cdots, x_n)) \leqslant 2^n$ ([19, p. 97]). One has the following sharper bound for $p(F)$ if F is a formally real field of transcendence degree d over a subfield, namely $p(F) \geqslant d + 1$ ([3]).

A major open question in this area is the following.

Question 5.7. *What is* $p(\mathbb{R}(x_1, x_2, x_3))$*?*

More on the arithmetic side, interesting questions arise concerning the Pythagoras number of a formally real function field over the field of rational numbers. The inequality $p(\mathbb{Q}(x)) \leqslant 8$ goes back to Landau (1906) and we have $p(k(x)) = p(k)+1 \leqslant 5$ for real number fields k ([19, Theorem 1.9, p. 100]). Conjecturally for a function field F in one variable over a number field, $p(F) \leqslant 5$ ([19, Conjecture 1.10, p. 100]). The estimate $p(F) \leqslant 7$ is due to Colliot-Thélène and a sharper estimate $p(F) \leqslant 6$ is due to Pop ([20]). One has the following general result for a function field F in d variables over a number field, namely $p(F) \leqslant 2^{d+1}$ (cf. [19, p. 100/101]). This combines the Milnor conjecture and a local-global principle for the Galois cohomology of higher dimensional function fields due to Jannsen in [8].

6. Dimensions of quadratic forms

In this section we discuss results on the dimension of quadratic forms satisfying certain conditions.

6.1. **Possible dimensions of anisotropic quadratic forms in $I^n(F)$.** An anisotropic n-fold Pfister form over F belongs to $I^n(F)$ and has dimension 2^n. By an Arason-Pfister theorem, every nonzero anisotropic form in $I^n(F)$ is of dimension at least 2^n. Are there other restrictions on the dimensions of anisotropic quadratic forms in $I^n(F)$?

Karpenko proved in [11] the following general result.

Theorem 6.1. *Let q be an anisotropic quadratic form such that $q \in I^n(F)$ for some $n \geqslant 1$. If $\dim(q) < 2^{n+1}$, then $\dim(q) = 2^{n+1} - 2^{i+1}$ for some $i \in \{0, 1, \ldots, n\}$. All other even degrees at least 2^{n+1} are possible.*

For example, the possible dimensions of anisotropic quadratic forms in $I^4(F)$ are

$$0, 16, 24, 28, 30, 32, 34, 36, \ldots.$$

6.2. **Hoffmann's Separation Theorem.** Let q and h be two anisotropic quadratic forms over F. When is q isotropic over the field $F(h)$? Equivalently, is there a rational morphism $X_h \dashrightarrow X_q$? If q is hyperbolic over $F(h)$, then $\dim(q) \geqslant \dim(h)$ ([13, Ch. X, Theorem 4.5]). This inequality does not hold in general if q is just isotropic over $F(h)$.

Example 6.2. Let h be a general n-fold Pfister form (i.e., h is similar to an n-fold Pfister form) and q a subform of h of dimension $> \frac{1}{2}\dim(h)$. Over the field $F(h)$ the Pfister form h is isotropic and hence hyperbolic. Therefore, q is isotropic over $F(h)$.

Note that the numbers $\dim(q)$ and $\dim(h)$ in the example are not separated by a power of 2. The following result proved by Hoffmann in [5] (known as the *separation theorem*) explains this observation.

Theorem 6.3. *Let q and h be two anisotropic quadratic forms over F. Suppose that $\dim(q) \leqslant 2^n < \dim(h)$ for some $n \geqslant 0$. Then q is anisotropic over $F(h)$.*

6.3. **The first Witt index.** Let q be an anisotropic quadratic form of dimension at least 2 over F. The form $q_{F(q)}$ is isotropic. Its Witt index is then positive, denoted $i_1(q)$, and is called the *first Witt index of* q. The "typical" value of the first Witt index is equal to 1. If q is a "generic" form, e.g., $q = \langle x_1, x_2, \ldots, x_n \rangle$ over the field $F(x_1, x_2, \ldots, x_n)$ of rational functions, then $i_1(q) = 1$. On the other hand, if q is an anisotropic m-fold Pfister form, then $i_1(q) = 2^{m-1}$ since q is hyperbolic over $F(q)$.

All possible values of the first Witt index of all quadratic forms of given dimension n were determined by Karpenko in [9].

Theorem 6.4. *Let q be an anisotropic quadratic form of dimension n. Write $n-1$ in base 2:*

$$n - 1 = 2^{k_1} + 2^{k_2} + \cdots + 2^{k_r}$$

with $0 \leqslant k_1 < k_2 < \cdots < k_r$. *Then*

$$i_1(q) - 1 = 2^{k_1} + 2^{k_2} + \cdots + 2^{k_s}$$

for some $s = 0, 1, \ldots, r-1$.

In other words, $i_1(\varphi)-1$ is the remainder upon dividing $\dim(q)-1$ by some power of 2 less than $\dim(q)$. In fact, all values of $i_1(q)$ given in the theorem are attained by some forms over appropriate fields. In particular, the number of possible values of $i_1(q)$ is equal to the number of 1's in the base 2 expression of the integer $\dim(q)-1$.

Example 6.5. 1) If $\dim(q) = 2^m + 1$, then $i_1(q) = 1$. This can also be deduced from Hoffmann's Separation Theorem.
2) All possible values of the first Witt index of an anisotropic form of dimension 2^m are the 2-powers $1, 2, 2^2, \ldots, 2^{m-1}$. The largest value 2^{m-1} is the first Witt index of an m-fold Pfister form.

6.4. **Splitting patterns of quadratic forms.** Let q be a quadratic form of dimension n over F and let

$$i_0 < i_1 < \cdots < i_{h-1} < i_h = [n/2]$$

denote all Witt indices of quadratic forms q_L over all field extensions of L/F. The tuple $\mathrm{SP}(q) = (i_0, i_1, \ldots, i_{h-1}, i_h)$ of strictly increasing integers is called the *splitting pattern* of q. The smallest integer i_0 is the Witt index of q. If q is anisotropic, i.e., $i_0 = 0$, the integer i_1 is the first Witt index $i_1(q)$.

All possible splitting patterns of quadratic forms of small dimension are determined by Vishik in [21]. For example, the splitting patterns of anisotropic 9-dimensional forms are $(0,1,4)$ and $(0,1,2,3,4)$. For arbitrary n, it is unknown which splitting patterns of n-dimensional forms occur.

Example 6.6. (Excellent form) We employ the following inductive definition. An anisotropic quadratic form q of dimension n is called *excellent* if either $q = 0$ or there is an excellent form q' such that $\dim(q') < \dim(q)$ and $q \perp q'$ is a general Pfister form. Quadratic forms of dimension up to 3 are excellent. A form of dimension 4 is excellent if and only if it has trivial discriminant. The splitting pattern of an excellent form is determined inductively as follows: if in the definition, $\dim(q) + \dim(q') = 2^m$, then

$$\mathrm{SP}(q) = \{0\} \cup (\mathrm{SP}(q') + n - 2^{m-1}).$$

In particular, the splitting pattern of an excellent form q depends just on the dimension of q. For example, if q is an excellent form of dimension 25, then $\mathrm{SP}(q) = \{0, 9, 12\}$.

6.5. **Izhboldin dimension.** Let q be a quadratic form over F. Izhboldin defined an integer associated with the quadric X_q, called the *Izhboldin dimension*:

$$\dim_{Izh}(X_q) = \dim(X_q) - i_1(q) + 1 = \dim(q) - i_1(q) - 1,$$

where $i_1(q)$ is the first Witt index of q.

Example 6.7. Let h and q be as in Example 6.2. Then $\dim_{Izh}(X_h) = \dim_{Izh}(X_q) = 2^{n-1} - 1$. Note that q is isotropic over $F(h)$ and h is isotropic over $F(q)$.

Theorem 6.8. *([10]) Let q be a quadratic form and let Y be a complete (possibly singular) algebraic variety over F with all closed points of even degree and such that Y has a closed point of odd degree over $F(q)$ (this holds, for example, if Y has a point over $F(q)$). Then* $\dim_{Izh}(X_q) \leqslant \dim(Y)$ *and in the case* $\dim_{Izh}(X_q) = \dim(Y)$ *the form q is isotropic over $F(Y)$.*

The following corollary can be viewed as a variant of the Separation Theorem.

Corollary 6.9. *Let h and q be anisotropic quadratic forms. If h is isotropic over $F(q)$, then* $\dim_{Izh}(X_q) \leqslant \dim_{Izh}(X_h)$ *and in the case* $\dim_{Izh}(X_q) = \dim_{Izh}(X_h)$ *the form q is isotropic over $F(h)$.*

7. Chow motives of quadrics

The constructions and results in this section are due to Vishik (see [21]). There is a functor from the category of smooth projective varieties to the additive category of *Chow Motives* taking a variety X to its motive $M(X)$ and a morphism $f : X \to Y$ of varieties to the class of the graph of f in the Chow group of the classes of algebraic cycles in $X \times Y$. The motive of the projective space $\mathbb{P}^n$ decomposes into a direct sum $\mathbb{Z} \oplus \mathbb{Z}(1) \oplus \cdots \oplus \mathbb{Z}(n)$, where $\mathbb{Z}(i)$ are the *Tate motives.*

Let X be a quadric of dimension n over an algebraically closed field. Then

$$M(X) \simeq \begin{cases} \mathbb{Z} \oplus \mathbb{Z}(1) \oplus \cdots \oplus \mathbb{Z}(n) = M(\mathbb{P}^n), & \text{if } n \text{ is odd;} \\ \mathbb{Z} \oplus \mathbb{Z}(1) \oplus \cdots \oplus \mathbb{Z}(k-1) \oplus \mathbb{Z}(k) \oplus \mathbb{Z}(k) \oplus \mathbb{Z}(k+1) \oplus \cdots \oplus \mathbb{Z}(n), & \text{if } n = 2k, \end{cases}$$

Note that the motive $\mathbb{Z}(k)$ appears in the decomposition twice if $n = 2k$.

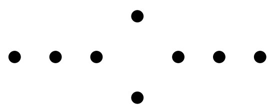
We introduce the set of symbols

$$\Lambda(X) = \Lambda(n) = \{0, 1, \ldots, k, \bar{k}, \ldots, \bar{1}, \bar{0}\},$$

where $k = [n/2]$ and a bijection between $\Lambda(X)$ and the set of Tate motives in the decomposition of $M(X)$ as follows: $i \leftrightarrow \mathbb{Z}(i)$ and $\bar{i} \leftrightarrow \mathbb{Z}(n-i)$ for $i = 0, 1, \ldots, k$.

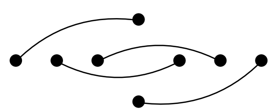
Over an arbitrary field the motive $M(X)$ is a direct sum $M_1 \oplus M_2 \oplus \cdots M_s$ of indecomposable motives. The collection of indecomposable motives M_i in the direct sum is unique up to isomorphism. Over an algebraic closure of F, every motive M_i is the sum of certain Tate motives. Thus, the set $\Lambda(X)$ is the disjoint union of s subsets $\Lambda_1, \Lambda_2, \ldots, \Lambda_s$ so that for every i the elements of Λ_i correspond to the Tate motives in the decomposition of M_i over an algebraic closure. The motive $M(X)$ is indecomposable if and only if $s = 1$.

We will consider the diagrams (graphs) with vertices given by the set $\Lambda(X)$. A *connection* is an edge in the diagram such that both ends of the edge belong to the same subset Λ_i. (But we don't necessarily connect every two vertices in the same subset Λ_i.) We will draw vertices horizontally in the order they appear in the definition of $\Lambda(X)$ with one exception: if n is even we draw the vertices k and $\bar{k}$ one under the other to indicate that they correspond to the same Tate motive $\mathbb{Z}(k)$. Below are the diagrams for a 6-dimensional hyperbolic (respectively, 7-dimensional generic) quadrics:

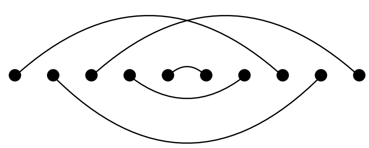
 

Example 7.1. Let q be a subform of dimension $2^{m-1}+1$ of a general anisotropic m-fold Pfister form h. Rost proved that there is a direct summand M of the motive $M(X_q)$ such that over an algebraic closure M is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}(2^{m-1}-1)$. This motive depends on h only (not on the choice of q), it is called the *Rost motive* of h and denoted M_h.

The motive of the Pfister quadric $M(X_h)$ is isomorphic to $M_h \oplus M_h(1) \oplus \cdots \oplus M_h(2^{m-1}-1)$. For example, if $m=3$, the diagram of $M(X_h)$ looks as follows:



Example 7.2. Let q be an excellent quadratic form of dimension $\geqslant 2$. The motive $M(X_q)$ is a direct sum of twists of Rost motives of the general Pfister forms appearing in the definition of an excellent form. For example, the diagram of the motive of the 11-dimensional excellent form $\langle\langle a,b,c\rangle\rangle \perp d\langle a,b,-ab\rangle$ is as follows:



The shape of the diagram depends only on $\dim(q)$. All connections in the diagram for an excellent form are called the *excellent connections.*

Let q be an anisotropic form with the splitting pattern $(i_0=0, i_1, \ldots, i_{h-1}, i_h)$ and $X = X_q$, $n = \dim(X)$. There are the following *standard* connections in the diagram of X.

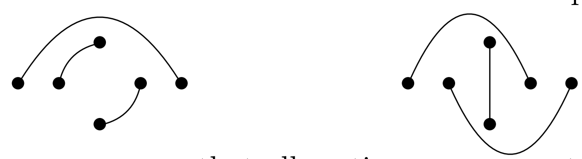
Proposition 7.3. *Let $i, j = 0, 1, \ldots, [n/2]$ be such that $i_{r-1} \leqslant i, j < i_r$, where $1 \leqslant r \leqslant h$, and $i + j = i_{r-1} + i_r - 1$. Then the symbols i and $\bar{j}$ are connected in the diagram of X.*

The standard connections for an excellent form coincide with excellent connections. Vishik proved in [23] that for an arbitrary form excellent connections also appear in the diagram:

Theorem 7.4. *Let q be an anisotropic quadratic form of dimension at least 2 and let f be an excellent form of the same dimension. Identify canonically $\Lambda(X_q)$ and $\Lambda(X_f)$. If two symbols are connected in $\Lambda(X_f)$, then they are also connected in $\Lambda(X_q)$.*

The theorem shows that the diagram of X_f is contained in the diagram of X_q. In particular, the diagram of an excellent form contains the smallest number of connections among the diagram of forms of the same dimension.

Example 7.5. Let q be an anisotropic *Albert form*, i.e., q is a 6-dimensional quadratic form with trivial discriminant. Its splitting pattern is $(0, 1, 3)$. Below are the two diagrams with the standard and excellent connections respectively.



Combining these diagrams we see that all vertices are connected, hence the motive $M(X_q)$ is indecomposable.

Indecomposability of the motive of a quadric is used in the following application due to Izhboldin and Karpenko in [12].

Theorem 7.6. *Let q and h be two anisotropic quadratic forms of the same odd dimension. Suppose that q is isotropic over $F(h)$ and h is isotropic over $F(q)$. If in addition at least one of the two motives $M(X_q)$ and $M(X_h)$ is indecomposable, then the forms q and h are similar.*

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