ON A PAIRING FOR ALGEBRAIC TORI

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Abstract. Let T be an algebraic torus over a field F. There is a pairing between the groups of torsors for the torus T and its dual with values in the third Galois cohomology group over all field extensions of F. We study the kernel of this pairing.

1. Introduction

For a field F there is a pairing

$$F^{\times} \otimes \mathrm{Br}(F) \to H^3(F, \mathbb{Q}/\mathbb{Z}(2)), \quad a \otimes v \mapsto a \cup v$$

where $\mathrm{Br}(F) = H^2(F, \mathbb{Q}/\mathbb{Z}(1))$ is the Brauer group of F.

Let L/F be a finite field extension, $x \in L^{\times}$ and $v \in \mathrm{Br}(L/F)$ an element of the Brauer group of F that is split by L. By the projection formula,

$$N_{L/F}(x) \cup v = N_{L/F}(x \cup v_L) = 0 \quad \text{in} \quad H^3(F, \mathbb{Q}/\mathbb{Z}(2)),$$

where $N_{L/F}$ is the norm (corestriction) homomorphism.

Conversely, let $a \in F^{\times}$ be such that $a \cup v = 0$ for all $v \in \mathrm{Br}(L/F)$. Is it true that $a = N_{L/F}(x)$ for some $x \in L^{\times}$? The answer is "no" if, for example, F is a totally imaginary number field since $H^3(F, \mathbb{Q}/\mathbb{Z}(2)) = 0$, but the norm map $N_{L/F}$ is not surjective for a nontrivial field extension L/F.

We can modify the question as follows. Let $a \in F^{\times}$ be such that $a \cup v = 0$ in $H^3(K, \mathbb{Q}/\mathbb{Z}(2))$ for all $v \in \mathrm{Br}(K \otimes L/K)$ and all field extensions K/F. Is it true that a is the norm in the field extension L/F?

The answer is positive if the extension L/F is so that the relative Brauer group $\mathrm{Br}(K \otimes L/K)$ can be "rationally parameterized". For example, if L/F is cyclic, every element in $\mathrm{Br}(K \otimes L/K)$ is represented by a cyclic simple K-algebra $(K \otimes L/K, t)$, where $t \in K^{\times}$ (thus, the group $\mathrm{Br}(K \otimes L/K)$ is parameterized by t). Now take $K = F(t)$ the rational function field in the variable t. If $a \cup (K \otimes L/K, t) = 0$ in $H^3(K, \mathbb{Q}/\mathbb{Z}(2))$, then taking the residue homomorphism with values in $H^2(F, \mathbb{Q}/\mathbb{Z}(1)) = \mathrm{Br}(F)$ with respect to the discrete valuation on K given by the parameter t, we get that the class of the cyclic algebra $(L/F, a)$ in $\mathrm{Br}(F)$ is trivial and hence a is the norm in L/F.

For an arbitrary finite field extension L/F we don't know how to answer the question. But if L/F is a Galois field extension, we show in the paper that the answer is positive.

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In fact, we will consider a more general problem. Let T be an algebraic torus over F. Write T° for the dual torus. For a field extension K/F, there is a pairing (see [1])

$$H^1(K,T^\circ)\otimes H^1(K,T)\to H^3(K,\mathbb{Q}/\mathbb{Z}(2)),\quad u\otimes v\mapsto u\cup v.$$

For example, if $T=R_{L/F}(\mathbb{G}_{m,L})/\,\mathbb{G}_m$ for a finite separable field extension L/F (here $R_{L/F}$ is the Weil transfer functor) and $K=F$ we get the pairing between $F^\times/N_{L/F}(L^\times)$ and $\mathrm{Br}(L/F)$ as above.

The *kernel* of the pairing is the subgroup of all $u\in H^1(F,T^\circ)$ such that $u_K\cup v=0$ in $H^3(K,\mathbb{Q}/\mathbb{Z}(2))$ for all $v\in H^1(K,T)$ and all field extensions K/F. The question is whether the kernel is trivial.

We don't know the answer to this question in general. In the paper we find certain classes of tori T such that the kernel of the pairing is trivial. For example, the kernel is trivial if isomorphism classes of T-torsors over field extensions K/F (i.e., the group $H^1(K,T)$) can be "rationally parameterized". This is the case when the classifying space $\mathrm{B}T$ is retract rational (see [8, Theorem 5.8.]). In Corollary 5.4 we show more generally that the kernel of the pairing is trivial if $\mathrm{B}T$ is 2-retract rational (for the definition see [8]). The latter is equivalent to 2-retract rationality of the dual torus T° by Theorem 7.3.

Let Π be the Galois group of the splitting field E of a torus T over F. The character group $\widehat{T}$ of T over E is a Π-lattice. The torus T is determined by the field extension E/F and the Π-lattice $\widehat{T}$. In the first part of the paper we define a finite abelian elementary 2-group $\Phi(\Pi,M)$ for every finite group Π and a Π-lattice M such that for every torus T with character Π-lattice $\widehat{T}=M$ there is a surjective homomorphism from $\Phi(\Pi,M)$ to the kernel of the pairing for T (Proposition 5.1). We also show that for every Π-lattice M this surjective map is an isomorphism for a "versal" torus T (Proposition 5.6). Thus the study of the kernel of the pairing reduces to the study of the group $\Phi(\Pi,M)$.

We don't know whether the group $\Phi(\Pi,M)$ can be nontrivial. It is shown in the paper that $\Phi(\Pi,M)$ is zero for certain classes of lattices (see Proposition 2.1). In Section 6 we give several examples of tori T with the trivial kernel of the pairing. Note that in these examples $\mathrm{B}T$ is not 2-retract rational.

The kernel of the pairing for a torus T is isomorphic to the torsion subgroup of the second Chow group $\mathrm{CH}^2(\mathrm{B}T)$ of the classifying space of T. Therefore, the triviality of the group $\Phi(\Pi,\widehat{T})$ implies $\mathrm{CH}^2(\mathrm{B}T)_{\mathrm{tors}}=0$.

In the appendix we present some results on p-retract rationality of algebraic tori and their classifying spaces.

The field F in the paper is arbitrary. If $\mathrm{char}(F)=p>0$, the definition of the p-component of the cohomology groups $H^{i+1}(F,\mathbb{Q}/\mathbb{Z}(i))$ requires extra care (see, for instance [5, Part 2, Appendix A]).

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2. Lattices

Let Π be a finite group acting on a lattice N. Define $\mathit{\Gamma}^2(N)$ as the factor group of $N \otimes N$ by the subgroup generated by $x \otimes y + y \otimes x$. We write $x \star y$ for the coset of $x \otimes y$. There is an exact sequence of Π-modules

$$0 \to N/2 \to \mathit{\Gamma}^2(N) \to \mathit{\Lambda}^2(N) \to 0, \tag{2.1}$$

where $\mathit{\Lambda}^2(N)$ is the second exterior power of N, the first map takes x to $x \star x$ and the second map takes $x \star y$ to $x \wedge y$.

Write

$$\alpha_N : H^1(\Pi, \mathit{\Lambda}^2(N)) \to H^2(\Pi, N/2)$$

for the connecting homomorphism for the exact sequence (2.1).

If P is a *permutation* Π-lattice with $\mathbb{Z}$-basis X, then the homomorphism $\mathit{\Gamma}^2(P) \to P/2$ taking $x \star x'$ to 0 if $x \neq x'$ and $x \star x$ to $x + 2P$ for $x, x' \in X$, is a splitting of the sequence (2.1) with $N = P$. In particular, $\alpha_P = 0$.

If N' is another Π-lattice, we have $\mathit{\Lambda}^2(N \oplus N') = \mathit{\Lambda}^2(N) \oplus (N \otimes N') \oplus \mathit{\Lambda}^2(N')$ and a similar formula holds for $\mathit{\Gamma}^2(N \oplus N')$. It follows that

$$\alpha_{N \oplus N'} = \alpha_N \oplus 0 \oplus \alpha_{N'}.$$

In particular, $\mathrm{Im}(\alpha_{N \oplus N'}) = \mathrm{Im}(\alpha_N) \oplus \mathrm{Im}(\alpha_{N'})$. If P is a permutation lattice, then $\mathrm{Im}(\alpha_{N \oplus P}) = \mathrm{Im}(\alpha_N)$.

Recall that two lattices N and N' are *stably equivalent* if $N \oplus P \simeq N' \oplus P'$ for some permutation lattices P and P'. If N and N' are stably equivalent, then $\mathrm{Im}(\alpha_N) \simeq \mathrm{Im}(\alpha_{N'})$.

Let M be a Π-lattice. Consider a *coflasque resolution*

$$0 \to N \to P \to M \to 0 \tag{2.2}$$

of M, where P is a permutation lattice and N is a coflasque lattice, i.e., $H^1(\Gamma, N) = 0$ for every subgroup $\Gamma \subset \Pi$ (see [2, Lemme 3]). Recall that N is uniquely determined by M up to stable equivalence by [2, Lemme 5]. It follows that the group $\mathrm{Im}(\alpha_N)$ is independent up to canonical isomorphism of the choice of the resolution of M. We set

$$\boxed{\Phi(\Pi, M) := \mathrm{Im}(\alpha_N).}$$

It is still unclear whether the group $\Phi(\Pi, M)$ is always trivial. Below we collect some properties of this group. A Π-lattice N is 2-*invertible* if there is an odd integer n such that the endomorphism of N of multiplication by n factors as $N \to P \to N$ for a permutation lattice P (see Appendix).

Proposition 2.1. *Let M be a Π-lattice. Then $\Phi(\Pi, M)$ is a finite group such that $2 \cdot \Phi(\Pi, M) = 0$. The group $\Phi(\Pi, M)$ is trivial in the following cases:*

(1) *The lattice N in the coflasque resolution (2.2) is 2-invertible,*
(2) *Sylow 2-subgroups of Π are cyclic or Klein four-groups.*

Proof. The first statement follows from the fact that the group $H^2(\Pi, N/2)$ is finite and 2-torsion. If N is 2-invertible, then $n\alpha_N = 0$ for an odd integer n, hence $\alpha_N = 0$. If Π is a cyclic group, $\Phi(\Pi, M)$ is trivial since every coflasque lattice is invertible (a direct summand of a permutation lattice) by [2, Proposition 2] and hence is 2-invertible. If $\Pi' \subset \Pi$ is a subgroup, we have the restriction and corestriction homomorphisms

$$\Phi(\Pi, M) \xrightarrow{\text{res}} \Phi(\Pi', M) \xrightarrow{\text{cor}} \Phi(\Pi, M)$$

with the composition multiplication by the index $[\Pi : \Pi']$. Therefore, if $[\Pi : \Pi']$ is odd, the restriction homomorphism is injective. Taking for Π' a Sylow 2-subgroup of Π, we see that $\Phi(\Pi, M)$ is trivial if Π' is cyclic. The case when Π' is a Klein four-group will be considered in Example 4.3. □

3. A filtration on $H^1(\Pi, M/2)$

Let M be a Π-lattice with a coflasque resolution (2.2). We define a two-term filtration on $H^1(\Pi, M/2)$ as follows. Set

$$\begin{aligned} H^1(\Pi, M/2)^{(1)} :&= \mathrm{Ker}\,[H^1(\Pi, M/2) \to H^2(\Pi, N/2)] \\ &= \mathrm{Im}\,[H^1(\Pi, P/2) \to H^1(\Pi, M/2)]. \end{aligned}$$

The following is an intrinsic description of $H^1(\Pi, M/2)^{(1)}$ showing that it does not depend on the choice of the resolution.

Lemma 3.1. *The subgroup $H^1(\Pi, M/2)^{(1)}$ is generated by the images of the compositions*

$$M^\Gamma \otimes H^1(\Gamma, \mathbb{Z}/2) \xrightarrow{\cup} H^1(\Gamma, M/2) \xrightarrow{\text{cor}} H^1(\Pi, M/2)$$

over all subgroups $\Gamma \subset \Pi$.

Proof. The permutation module P is a direct sum of modules of the form $\mathbb{Z}[\Pi/\Gamma]$, where Γ is a subgroup of Π. A homomorphism $\mathbb{Z}[\Pi/\Gamma] \to M$ of Π-modules is determined by the image $m \in M^\Gamma$ of the coset of 1. The image of the induced homomorphism

$$H^1(\Gamma, \mathbb{Z}/2) = H^1(\Pi, \mathbb{Z}/2[\Pi/\Gamma]) \to H^1(\Pi, M/2)$$

takes $x \in H^1(\Gamma, \mathbb{Z}/2)$ to the image of $m \otimes x$ under the composition in the statement of the lemma.

Conversely, every Π-module homomorphism $\mathbb{Z}[\Pi/\Gamma] \to M$ factors into a composition $\mathbb{Z}[\Pi/\Gamma] \to P \to M$ since $\mathrm{Ext}^1_\Pi(\mathbb{Z}[\Pi/\Gamma], N) = H^1(\Gamma, N) = 0$ as N is coflasque. □

Recall that the choice of a permutation $\mathbb{Z}$-basis $\{x_i\}$ of P yields a homomorphism $\mathit{\Gamma}^2(P) \to P/2$ and hence its restriction $j : \mathit{\Gamma}^2(N) \to P/2$. We claim that the composition $N/2 \to \mathit{\Gamma}^2(N) \xrightarrow{j} P/2$ coincides with the natural embedding. Indeed, the composition takes $\sum a_i x_i$ to

$$j\big[\big(\textstyle\sum a_i x_i\big) \star \big(\textstyle\sum a_i x_i\big)\big] = \textstyle\sum a_i^2 x_i = \textstyle\sum a_i x_i.$$

We get a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & N/2 & \longrightarrow & \mathit{\Gamma}^2(N) & \longrightarrow & \mathit{\Lambda}^2(N) & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle j} & & \downarrow{\scriptstyle k} & & \\ 0 & \longrightarrow & N/2 & \longrightarrow & P/2 & \longrightarrow & M/2 & \longrightarrow & 0. \end{array}$$

Note that k depends on the choice of a basis in P.

It follows that the homomorphism α_N factors as follows:

$$\alpha_N : H^1(\Pi, \mathit{\Lambda}^2(N)) \xrightarrow{\beta_N} H^1(\Pi, M/2) \to H^2(\Pi, N/2),$$

where $\beta_N = k^*$. The kernel of the last homomorphism is equal to $H^1(\Pi, M/2)^{(1)}$.

Let $M^\circ = \mathrm{Hom}(M, \mathbb{Z})$ be the dual lattice.

Denote by $H^1(\Pi, M/2)^{(2)}$ the subgroup of all elements $u \in H^1(\Pi, M/2)$ such that $\mathrm{res}_{\Pi/\Gamma}(u) \cup y = 0$ in $H^2(\Gamma, \mathbb{Z}/2)$ for all $y \in H^1(\Gamma, M^\circ)$ and all subgroups $\Gamma \subset \Pi$. The cup-product is taken for the pairing

$$H^1(\Gamma, M/2) \otimes H^1(\Gamma, M^\circ) \to H^2(\Gamma, \mathbb{Z}/2).$$

For a subgroup $\Gamma \subset \Pi$, the connecting homomorphism

$$\partial : (N^\circ)^\Gamma = H^0(\Gamma, N^\circ) \to H^1(\Gamma, M^\circ)$$

induced by the exact sequence

$$0 \to M^\circ \to P^\circ \to N^\circ \to 0$$

is surjective since $H^1(\Pi, P^\circ) = 0$. For an element $y \in H^1(\Gamma, M^\circ)$ choose an $x \in (N^\circ)^\Gamma$ such that $y = \partial(x)$. The composition

$$H^1(\Gamma, M/2) \to H^2(\Gamma, N/2) \xrightarrow{x^*} H^2(\Gamma, \mathbb{Z}/2)$$

is given by the cup-product with $y \in H^1(\Gamma, M^\circ)$. It follows that

$$H^1(\Pi, M/2)^{(1)} \subset H^1(\Pi, M/2)^{(2)}.$$

We also have a commutative diagram

$$\begin{array}{ccc} H^1(\Gamma, \mathit{\Lambda}^2(N)) & \xrightarrow{\alpha_N} & H^2(\Gamma, N/2) \\ \downarrow{\scriptstyle x^*} & & \downarrow{\scriptstyle x^*} \\ H^1(\Gamma, \mathit{\Lambda}^2(\mathbb{Z})) & \xrightarrow{\alpha_{\mathbb{Z}}} & H^2(\Gamma, \mathbb{Z}/2). \end{array}$$

As $\mathit{\Lambda}^2(\mathbb{Z}) = 0$, the composition $x^* \circ \alpha_N$ is trivial.

Recall that α_N is the composition of β_N and the connecting homomorphism $H^1(\Pi, M/2) \to H^2(\Pi, N/2)$. We have proved:

Proposition 3.2. *The image of the homomorphism* $\beta_N : H^1(\Pi, \mathit{\Lambda}^2(N)) \to H^1(\Pi, M/2)$ *is contained in* $H^1(\Pi, M/2)^{(2)}$.

Thus, $\Phi(\Pi, M)$ is the image of the composition

$$H^1(\Pi, \Lambda^2(N)) \to H^1(\Pi, M/2)^{(2/1)} \hookrightarrow H^2(\Pi, N/2).$$

Note that although β_N does depend on the choice of a basis in P, the first map in the composition is independent of the choice as is the composition.

Corollary 3.3. *If $H^1(\Pi, M/2)^{(1)} = H^1(\Pi, M/2)^{(2)}$, then $\Phi(\Pi, M) = 0$.*

4. Examples

In this section we consider several classes of Π-lattices M with $\Phi(\Pi, M) = 0$.

Let a finite group Π act on a finite set X and let I be the kernel of the augmentation map $\mathbb{Z}[X] \to \mathbb{Z}$ taking every x in X to 1. The exact sequence

$$0 \to I/2 \to (\mathbb{Z}/2)[X] \to \mathbb{Z}/2 \to 0$$

yields a connecting homomorphism $\partial : \mathbb{Z}/2 \to H^1(\Pi, I/2)$. Write

$$t_X := \partial(1 + 2\mathbb{Z}) \in H^1(\Pi, I/2).$$

Lemma 4.1. *If t_X is not trivial and there is an element $\sigma \in \Pi$ of order 2 without fixed points in X then $t_X \notin H^1(\Pi, I/2)^{(2)}$.*

Proof. Let Γ be cyclic subgroup of Π (of order 2) generated by σ. The exact sequence

$$0 \to \mathbb{Z} \to \mathbb{Z}[X] \to I^\circ \to 0$$

yields an exact sequence

$$0 = H^1(\Gamma, \mathbb{Z}[X]) \to H^1(\Gamma, I^\circ) \xrightarrow{\lambda} H^2(\Gamma, \mathbb{Z}) \to H^2(\Gamma, \mathbb{Z}[X]).$$

By assumption, $\mathbb{Z}[X]$ is a free Γ-module, hence $H^2(\Gamma, \mathbb{Z}[X]) = 0$. Therefore, the map λ is an isomorphism. We have the following diagram of cup-product maps:

$$\begin{array}{ccccc}
[\ \mathbb{Z}/2 & \otimes & H^2(\Gamma, \mathbb{Z})\ ] & \xrightarrow{\cup} & H^2(\Gamma, \mathbb{Z}/2) \\
\Big\downarrow{\scriptstyle \mathrm{res}\circ\partial} & & \Big\uparrow{\scriptstyle \lambda\ \wr} & & \Big\| \\
[\ H^1(\Gamma, I/2) & \otimes & H^1(\Gamma, I^\circ)\ ] & \xrightarrow{\cup} & H^2(\Gamma, \mathbb{Z}/2)
\end{array}$$

Let $y \in H^1(\Gamma, I^\circ)$ be (the only) nonzero element. Then the element

$$\mathrm{res}(t_X) \cup y = \bar{1} \cup \lambda(y)$$

is the image of nonzero $\lambda(y)$ under the isomorphism $H^2(\Gamma, \mathbb{Z}) \simeq H^2(\Gamma, \mathbb{Z}/2)$ hence $\mathrm{res}(t_X) \cup y \neq 0$. It follows that $t_X \notin H^1(\Pi, I/2)^{(2)}$. ☐

Example 4.2. Let I be the kernel of the augmentation map $\mathbb{Z}[\Pi] \to \mathbb{Z}$. We claim that the group $H^1(\Pi, I/2)^{(2)}$ is trivial, in particular $\Phi(\Pi, I) = 0$ by Proposition 3.2. We may assume that the order of Π is even. Let $\sigma \in \Pi$ be an element of order 2. As σ act without fixed points by left translations on $X = \Pi$, by the Lemma 4.1, $t_X \notin H^1(\Pi, I/2)^{(2)}$. On the other hand, the group $H^1(\Pi, I/2) \simeq \mathbb{Z}/2$ is generated by t_X. It follows that $H^1(\Pi, I/2)^{(2)}$ is trivial.

Example 4.3. Let Π be the Klein group of order 4. We show that $\Phi(\Pi, M) = 0$ for every Π-lattice M. In Example 4.2 we proved that $\Phi(\Pi, I) = 0$, where I is the augmentation ideal in $\mathbb{Z}[\Pi]$. Let N be a coflasque module in a coflasque resolution of I. Then the map α_N is trivial.

By a theorem of Kunyavskiĭ [6], every coflasque Π-module is stably equivalent to the direct sum of several copies of N. Therefore, $\alpha_{N'} = 0$ for every coflasque N' and hence $\Phi(\Pi, M) = 0$ for every Π-lattice M.

Example 4.4. Let Π be an elementary abelian 2-group of order 2^n. Choose subgroups $\Pi_1, \Pi_2, \ldots, \Pi_n$ in Π of index 2 with zero intersection. The group Π acts naturally on the set X the disjoint union of Π/Π_i over all i. Let I be the kernel of $\mathbb{Z}[X] \to \mathbb{Z}$ as in Lemma 4.1. We claim that $\Phi(\Pi, I) = 0$.

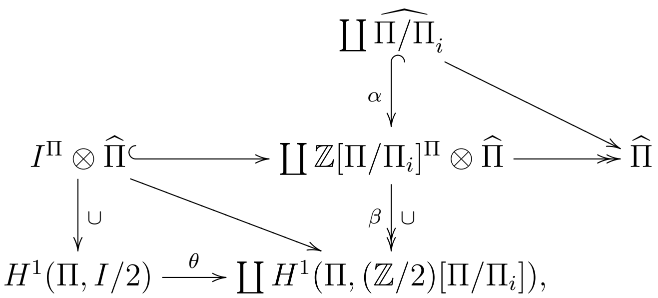
Consider the following exact sequence of cohomology groups:

$$0 \to \mathbb{Z}/2 \to H^1(\Pi, I/2) \xrightarrow{\theta} \textstyle\coprod_{i=1}^{n} H^1(\Pi, (\mathbb{Z}/2)[\Pi/\Pi_i]). \tag{4.1}$$

Let x_i be a generator of the infinite cyclic group $\mathbb{Z}[\Pi/\Pi_i]^{\Pi}$. We have an exact sequence

$$0 \to I^{\Pi} \to \textstyle\coprod_{i=1}^{n} \mathbb{Z}[\Pi/\Pi_i]^{\Pi} \to \mathbb{Z} \to 0,$$

where the image of every x_i is equal to 1. Therefore, there is a commutative diagram with the exact row:



where α takes a character $\chi : \Pi/\Pi_i \to \mathbb{Z}/2$ to $x_i \otimes \chi'$, where χ' is the composition of $\Pi \to \Pi/\Pi_i$ and χ. If we identify $H^1(\Pi, (\mathbb{Z}/2)[\Pi/\Pi_i])$ with $\widehat{\Pi}_i$, the map β takes $x_i \otimes \chi$ to the restriction of χ to Π_i. It follows that the column in the diagram is exact.

As the map $\coprod \widehat{\Pi/\Pi_i} \to \widehat{\Pi}$ is an isomorphism, by diagram chase, the other diagonal map in the diagram is also isomorphism. It follows that the restriction of the map θ in (4.1) on $H^1(\Pi, I/2)^{(1)}$ is surjective. Therefore, the group $H^1(\Pi, I/2)$ is generated by $H^1(\Pi, I/2)^{(1)}$ and the image t of $1 + 2\mathbb{Z}$ under the map $\mathbb{Z}/2 \to H^1(\Pi, I/2)$.

Let $\sigma \in \Pi$ be (the only) element that is not contained in Π_i for all i. Then σ acts without fixed points on each set Π/Π_i. By Lemma 4.1, $t \notin H^1(\Pi, I/2)^{(2)}$. It follows that $H^1(\Pi, I/2)^{(1)} = H^1(\Pi, I/2)^{(2)}$. In view of Corollary 3.3, $\Phi(\Pi, I) = 0$.

5. Algebraic tori

Let T be an algebraic torus over a field F, let E/F be a splitting field of T and $\Pi = \mathrm{Gal}(L/F)$. Write $\widehat{T}$ for the character Π-lattice of T over E.

Denote by T° the dual torus. The character lattice of T° is dual to $\widehat{T}$.

For a field extension K/F, there is a pairing (see [1]):

$$H^1(K,T^\circ)\otimes H^1(K,T)\to H^3(K,\mathbb{Q}/\mathbb{Z}(2)),\quad u\otimes v\mapsto u\cup v. \tag{5.1}$$

The *kernel* of the pairing is the subgroup of all $u\in H^1(F,T^\circ)$ such that $u_K\cup v=0$ in $H^3(K,\mathbb{Q}/\mathbb{Z}(2))$ for all $v\in H^1(K,T)$ and all field extensions K/F.

There are two descriptions of the kernel of the pairing. First, the kernel is canonically isomorphic to the torsion part of $\mathrm{CH}^2(\mathrm{B}T)$, where $\mathrm{B}T$ is the classifying space of T (see [1, Theorem B]).

The second description is as follows.

Proposition 5.1. *There is a natural homomorphism $\Phi(\Pi,\widehat{T})\to H^1(F,T^\circ)$ with image the kernel of the pairing (5.1) for the torus T.*

Proof. For any torus S split by E denote by q_S the homomorphism

$$\widehat{S}/2\to\widehat{S}\otimes E^\times = S^\circ(E),$$

taking x to $x\otimes(-1)$.

Every character $x\in\widehat{S}$ can be viewed as an invertible function on S which we denote by e^x.

Let

$$1\to T\to R\to Q\to 1$$

be a coflasque resolution for T of tori that are split by E.

For any point $x\in Q_E := Q\times_F \mathrm{Spec}(E)$ of codimension 1, there is *residue* homomorphism

$$\partial_x : K_2(E(Q))\to E(x)^\times$$

from Milnor's K_2-group of the function field of Q over E to the multiplicative group of the residue field $E(x)$. We write $\overline{A}^0(Q_E,\mathcal{K}_2)$ for the factor group of the intersection of the kernels of ∂_x over all points $x\in Q_E$ of codimension 1 by the subgroup $K_2(E)$. There is an exact sequence (see [5, §5.7])

$$0\to Q^\circ(E)\to\overline{A}^0(Q_E,\mathcal{K}_2)\to\Lambda^2(\widehat{Q})\to 0.$$

The first map takes $x\otimes a\in\widehat{Q}\otimes E^\times = Q^\circ(E)$ to the symbol $\{e^x,a\}$ and the second map takes a symbol $\{e^x,e^y\}$ to $x\wedge y$.

We also have a homomorphism

$$p:\Gamma^2(\widehat{Q})\to\overline{A}^0(Q_E,\mathcal{K}_2),$$

taking $x\star y$ to the symbol $\{e^x,e^y\}$.

We have the following commutative diagram with the exact rows:

$$\begin{array}{ccccc}
Q^\circ(E) & \longrightarrow & \overline{A}^0(Q_E,\mathcal{K}_2) & \longrightarrow & \Lambda^2(\widehat{Q}) \\
{\scriptstyle q_Q}\uparrow & & {\scriptstyle p}\uparrow & & \| \\
\widehat{Q}/2 & \longrightarrow & \Gamma^2(\widehat{Q}) & \longrightarrow & \Lambda^2(\widehat{Q}) \\
\| & & {\scriptstyle j}\downarrow & & {\scriptstyle k}\downarrow \\
\widehat{Q}/2 & \longrightarrow & \widehat{R}/2 & \longrightarrow & \widehat{T}/2 \\
{\scriptstyle q_Q}\downarrow & & {\scriptstyle q_R}\downarrow & & {\scriptstyle q_T}\downarrow \\
Q^\circ(E) & \longrightarrow & R^\circ(E) & \longrightarrow & T^\circ(E),
\end{array}$$

where j and k were defined in Section 3. Therefore, the following diagram

$$\begin{array}{ccccc}
H^1(\Pi,\Lambda^2(\widehat{Q})) & \xrightarrow{\beta_{\widehat{Q}}} & H^1(\Pi,\widehat{T}/2) & \longrightarrow & H^2(\Pi,\widehat{Q}/2) \\
& \searrow & {\scriptstyle q_T^*}\downarrow & & {\scriptstyle q_Q^*}\downarrow \\
& & H^1(\Pi,T^\circ(E)) & \hookrightarrow & H^2(\Pi,Q^\circ(E)),
\end{array} \tag{5.2}$$

is commutative and the composition

$$H^1(\Pi,\Lambda^2(\widehat{Q})) \to H^1(\Pi,T^\circ(E)) \hookrightarrow H^2(\Pi,Q^\circ(E))$$

is the connecting map for the top exact sequence in the diagram above. Note that the second map in the composition is injective since $H^1(\Pi,R^\circ(E))=0$ as R is a quasi-split torus.

It was proven in [1, Theorem 4.7] that the image of the composition and the image in $H^2(\Pi,Q^\circ(E))$ of the kernel of the pairing coincide. Note that although the map k depends on the choice of a basis in $\widehat{R}$, the composition does not. Therefore, the homomorphism $H^1(\Pi,\Lambda^2(\widehat{Q}))\to H^1(\Pi,T^\circ(E))$ also does not depend on the choice of a basis. Finally, the latter homomorphism factors into a composition of the natural surjection $H^1(\Pi,\Lambda^2(\widehat{Q}))\to\Phi(\Pi,\widehat{T})$ and the map $\Phi(\Pi,\widehat{T})\to H^1(\Pi,T^\circ(E))=H^1(F,T^\circ)$ with image the kernel of the pairing. □

Corollary 5.2. *The group* $\mathrm{CH}^2(\mathrm{B}T)_{\mathrm{tors}}$ *is canonically isomorphic to a factor group of* $\Phi(\Pi,\widehat{T})$.

Corollary 5.3. *If* $\Phi(\Pi,\widehat{T})=0$*, then* $\mathrm{CH}^2(\mathrm{B}T)_{\mathrm{tors}}=0$ *and the kernel of the pairing (5.1) is trivial.*

The following corollary is a consequence of Proposition 2.1, Corollary 5.3 and Theorem 7.3.

Corollary 5.4. *If* $\mathrm{B}T$ *is 2-retract rational, then* $\mathrm{CH}^2(\mathrm{B}T)_{\mathrm{tors}}=0$ *and the kernel of the pairing (5.1) is trivial.*

Corollary 5.5. *If* $\mathrm{char}(F) = 2$, *then* $\mathrm{CH}^2(\mathrm{B}T)_{\mathrm{tors}} = 0$ *and the kernel of the pairing (5.1) is trivial.*

Proof. The map $q_Q : \widehat{Q}/2 \to Q^\circ(E)$ in the proof of the proposition is trivial as $-1 = 1$. □

Proposition 5.1 shows that size of the kernel of the pairing is bounded by the size of the group $\Phi(\Pi, \widehat{T})$. We show that this bound is exact. In fact, for a "versal" torus T with the splitting group Π the homomorphism

$$\Phi(\Pi, \widehat{T}) \to H^1(F, T^\circ)$$

in Proposition 5.1 is injective. Indeed, consider a faithful representation $\Pi \to \mathrm{GL}(V)$ over $\mathbb{Q}$ and a versal Galois Π-extension $E := \mathbb{Q}(V)$ over the field $F := \mathbb{Q}(V)^\Pi$.

Proposition 5.6. *Let* M *be a* Π*-lattice and let* T *be a torus over* $F = \mathbb{Q}(V)^\Pi$ *with splitting field* E *and character lattice* $\widehat{T} = M$. *Then the homomorphism* $\Phi(\Pi, \widehat{T}) \to H^1(F, T^\circ)$ *is injective. In particular, the kernel of the pairing (5.1) for* T *is isomorphic to* $\Phi(\Pi, \widehat{T})$.

Proof. Consider a coflasque resolution (2.2) for M and a torus Q over F with splitting field E and character group N. In particular,

$$Q^\circ(E) = N \otimes \mathbb{Q}(V)^\times.$$

Tensoring with N the exact sequence

$$1 \to \mathbb{Q}^\times \to \mathbb{Q}(V)^\times \to D \to 0,$$

where D is the divisor Π-module of the affine space of V over $\mathbb{Q}$, we get an exact sequence of Π-modules:

$$1 \to N \otimes \mathbb{Q}^\times \to Q^\circ(E) \to N \otimes D \to 0.$$

As D is a permutation Π-module and N is coflasque, we have

$$H^1(\Pi, N \otimes D) = 0.$$

It follows that the natural homomorphism

$$H^2(\Pi, N \otimes \mathbb{Q}^\times) \to H^2(\Pi, Q^\circ(E)) \hookrightarrow H^2(F, Q^\circ)$$

is injective. As μ_2 is a direct factor of $\mathbb{Q}^\times$, the map

$$H^2(\Pi, N/2) \to H^2(\Pi, N \otimes \mathbb{Q}^\times)$$

is also injective. The statement follows the commutativity of the diagram (5.2) as $\Phi(\Pi, \widehat{T})$ is a subgroup of $H^2(\Pi, N/2)$. □

6. Examples of pairings

In this section we consider two applications.

Let L/F be a finite Galois field extension with Galois group Π and $T = R_{L/F}(\mathbb{G}_{m,L})/\,\mathbb{G}_m$. Then $T^\circ = R^{(1)}(\mathbb{G}_{m,L})$ is the torus of norm 1 elements in L. For a field extension K/F,

$$H^1(K,T) = \mathrm{Br}(KL/L) \quad \text{and} \quad H^1(K,T^\circ) = K^\times/N_{KL/K}(KL^\times),$$

where $KL := K \otimes L$.

The character lattice $\widehat{T}$ is the kernel I of the augmentation map $\mathbb{Z}[\Pi] \to \mathbb{Z}$. By Example 4.2, $\Phi(\Pi, I) = 0$. Therefore, Corollary 5.3 yields the following proposition.

Proposition 6.1. *Let L/F be a finite Galois field extension. Suppose that for an element $a \in F^\times$ we have $a \cup v = 0$ in $H^3(K, \mathbb{Q}/\mathbb{Z}(2))$ for all $v \in \mathrm{Br}(KL/K)$ and all field extensions K/F. Then a is the norm in the extension L/F.* □

Note that the torus T° (equivalently, BT) is 2-retract rational if and only if Sylow 2-subgroups of Π are metacyclic (see [4] or [10, §4.8, Theorem 3]).

Let $L_1, L_2, \dots, L_n$ be linearly disjoint separable quadratic field extension of a field F. Write L for the composite of all L_i and set $\Pi := \mathrm{Gal}(L/F)$.

Let T be the cokernel of the diagonal embedding

$$\mathbb{G}_m \hookrightarrow \textstyle\prod_{i=1}^n R_{L_i/F}(\mathbb{G}_{m,L_i}).$$

Let $\Pi_i := \mathrm{Gal}(L/L_i)$ and let X be the disjoint union of n Π-sets Π/Π_i. Then $\widehat{T} = I$ in the notation of Example 4.4. It was proved in that example that $\Phi(\Pi, I) = 0$.

For a field extension K/F, we have

$$H^1(K,T) = \bigcap_{i=1}^n \mathrm{Br}(KL_i/K).$$

From the exact sequence for the dual torus T°:

$$1 \to T^\circ \to \textstyle\prod_{i=1}^n R_{L_i/F}(\mathbb{G}_{m,L_i}) \to \mathbb{G}_m \to 1$$

we get

$$H^1(F,T^\circ) = F^\times/\textstyle\prod N_{L_i/F}(L_i^\times).$$

Then Corollary 5.3 yields:

Proposition 6.2. *Let $L_1, L_2, \dots, L_n$ be linearly disjoint separable quadratic field extension of a field F. Suppose that for an element $a \in F^\times$ we have $a \cup v = 0$ in $H^3(K, \mathbb{Q}/\mathbb{Z}(2))$ for and all $v \in \bigcap \mathrm{Br}(KL_i/K)$ and all field extensions K/F. Then a is the product of norms in the extensions L_i/F.* □

Note that the torus T° (equivalently, BT) is not 2-retract rational for $n \geq 3$.

7. Appendix

Let p be a prime integer. A Π-lattice M is called *p-invertible* if there is an integer n prime to p such that the endomorphism of multiplication by n of M can be factored as $M \to P \to M$, where P is a permutation lattice.

The following statement was proved in [9, Proposition 3.1]. For completeness, we give a slightly shorter prove below. For the definition of p-retract rationality see [8].

Proposition 7.1. *Let T be a torus over F, p a prime integer and $1 \to S \to R \to T \to 1$ a flasque resolution of T. Then T is p-retract rational over F if and only if $\widehat{S}$ is p-invertible.*

Proof. $\Rightarrow$: Let E/F be a splitting field of T with Galois group Π. For a smooth variety X over F set

$$\mathsf{U}(X) := E[X]^\times / E^\times.$$

Then $\mathsf{U}(X)$ is a Π-lattice. For example, $\mathsf{U}(T) = \widehat{T}$ (see [2, §2]).

If $\operatorname{Pic}(X_E) = 0$ and $W \subset X$ is a dense open subset, there is an exact sequence

$$0 \to \mathsf{U}(X) \to \mathsf{U}(W) \to P \to 0$$

for a permutation Π-lattice P (see [2, Proposition 5]).

As T is p-retract rational, there is a composition of morphisms of integral varieties $f : Z \to V \to W$, where V is an open subset of an affine space, W is an open subset of T and f is dominant of degree n prime to p. Shrinking the varieties we may assume that f is finite flat. We have a push-out commutative diagram of lattices

$$\begin{array}{ccccc}
\widehat{T} & \hookrightarrow & \widehat{R} & \twoheadrightarrow & \widehat{S} \\
\downarrow & & \downarrow & & \| \\
\mathsf{U}(W) & \hookrightarrow & P' & \twoheadrightarrow & \widehat{S} \\
\downarrow & & \downarrow & & \\
P & = & P & &
\end{array}$$

with the exact rows and columns with P a permutation lattice. As $\widehat{R}$ is permutation, the middle vertical sequence is split, hence P' is also permutation.

The push-forward (norm) homomorphism given by f yields a composition $\mathsf{U}(W) \xrightarrow{f^*} \mathsf{U}(Z) \xrightarrow{f_*} \mathsf{U}(W)$ that is multiplication by n. Since the map $f^* : \mathsf{U}(W) \to \mathsf{U}(Z)$ factors through the permutation lattice $\mathsf{U}(V)$, so does the endomorphism of $\mathsf{U}(W)$ of multiplication by n. As $\widehat{S}$ is flasque, we have $\operatorname{Ext}^1_\Pi(\widehat{S}, \mathsf{U}(V)) = 0$. It follows that the group $\operatorname{Ext}^1_\Pi(\widehat{S}, \mathsf{U}(W))$ is n-periodic.

Hence there is a diagram

$$\begin{array}{ccccccccc} & & & & & & \widehat{S} & & \\ & & & & & \swarrow & \downarrow{\scriptstyle n} & & \\ 0 & \longrightarrow & \boldsymbol{U}(W) & \longrightarrow & P' & \longrightarrow & \widehat{S} & \longrightarrow & 0, \end{array}$$

i.e., $\widehat{S}$ is p-invertible.

$\Leftarrow$: By assumption, the map $n : \widehat{S} \to \widehat{S}$ factors through a permutation lattice P for some n prime to p. As $H^1(F(T), P) = 1$, the group $H^1(F(T), S)$ is n-torsion, hence the pull-back of the sequence in the statement of the proposition with respect to the homomorphism $T \to T$ taking t to t^n is split generically, i.e., we have a commutative diagram

$$\begin{array}{ccccccccc} & & & & & & W & & \\ & & & & & \swarrow & \downarrow{\scriptstyle n} & & \\ 0 & \longrightarrow & S & \longrightarrow & R & \longrightarrow & T & \longrightarrow & 0, \end{array}$$

with $W \subset T$ a nonempty open subset. It follow that T is p-retract rational as R is a rational variety (see also [8, Remark 2.1]). $\square$

The following statement is a p-local analog of [3, Proposition 7.4].

Proposition 7.2. *Let S be a torus over F and p a prime integer. Then $\widehat{S}$ is p-invertible if and only if $H^1(K, S)$ has no element of order p for all field extensions K/F.*

Proof. $\Rightarrow$: There is an integer n prime to p such that the homomorphism $S \to S$ taking s to s^n factors through a quasi-split torus R'. Since $H^1(K, R') = 1$, the group $H^1(K, S)$ is n-periodic.

$\Leftarrow$: The order n of the generic S-torsor is prime to p. By [7, Theorem 2.2], there are subgroups $\Pi_i \subset \Pi$, $i = 1, 2, \dots, m$, characters $x_i \in \widehat{S}^{\Pi_i}$ and co-characters $y_i \in \operatorname{Hom}_{\Pi_i}(\widehat{S}, \mathbb{Z})$ such that

$$\textstyle\sum_i \operatorname{cor}_{\Pi/\Pi_i}(\varphi_i) = n \cdot 1_{\widehat{S}},$$

where φ_i is an endomorphism of $\widehat{S}$ defined by $\varphi_i(z) = y_i(z)x_i$.

Let $P = \coprod_i \mathbb{Z}[\Pi/\Pi_i]$. The elements x_i and y_i determine homomorphisms $f : P \to \widehat{S}$ and $g : \widehat{S} \to P$ such that $f \circ g = n \cdot 1_{\widehat{S}}$. By definition, $\widehat{S}$ is p-invertible. $\square$

Let T be an algebraic torus over F. Let

$$1 \to T \to R \to Q \to 1 \tag{7.1}$$

be an exact sequence of tori with R a quasi-split torus. We have $R = R_{C/F}(\mathbb{G}_{m,C})$ for an étale F-algebra C. Therefore, the variety of R is an open subscheme of the affine space $\mathbb{A}(C)$ where the torus T acts linearly. We can view Q as an "approximation" of the classifying space $\mathrm{B}T$. The p-retract rational type of Q is independent of the choice of the coflasque resolution of T. We say that $\mathrm{B}T$ is p-retract rational if so is Q.

The next statement is a p-local analog of [9, Proposirion 6.1].

Theorem 7.3. *Let T be a torus over F and p a prime integer. The following are equivalent:*

(1) $\mathrm{B}T$ *is p-retract rational,*
(2) T° *is p-retract rational,*
(3) *The group of R -equivalence classes $T^\circ(K)/R$ (see* [2, §5]*) has no element of order p for all field extensions K/F,*
(4) *If $1 \to T \to R \to Q \to 1$ is a coflasque resolution of T, then the lattice $\widehat{Q}$ is p-invertible.*

Proof. The equivalence of (2) and (4) is proved in Proposition 7.1 (with T replaced by T°).

(1) $\Rightarrow$ (4): As Q is an approximation of $\mathrm{B}T$, the torus Q is p-retract rational. Choose a flasque resolution $1 \to S \to R' \to Q \to 1$ of Q. In view of Proposition 7.1 applied to Q, the lattice $\widehat{S}$ is p-invertible. It follows that the group $\mathrm{Ext}^1_\Pi(\widehat{S}, \widehat{Q})$ is n-torsion for some integer n prime to p. Therefore we have a commutative diagram

$$\begin{array}{ccccccccc} & & \widehat{Q} & & & & & & \\ & & \uparrow{\scriptstyle n} & \nwarrow & & & & & \\ 0 & \longrightarrow & \widehat{Q} & \longrightarrow & \widehat{R}' & \longrightarrow & \widehat{S} & \longrightarrow & 0, \end{array}$$

hence $\widehat{Q}$ is p-invertible.

(4) $\Rightarrow$ (1): There is an integer n prime to p such that the map $n : \widehat{Q} \to \widehat{Q}$ factors through $\widehat{R}'$ for a quasi-split torus R'. thus we have a diagram

$$\begin{array}{ccc} & & Q \\ & \swarrow & \downarrow{\scriptstyle n} \\ R' & \longrightarrow & Q. \end{array}$$

It follows that Q and hence $\mathrm{B}T$ is p-retract rational as R' is a rational variety.

(3) $\Leftrightarrow$ (4): Dualising (7.1), we get a flasque resolution

$$1 \to Q^\circ \to R^\circ \to T^\circ \to 1$$

of Q°. By [2, Theorem 2], $T^\circ(K)/R \simeq H^1(K, Q^\circ)$. In view of Proposition 7.2, (3) is equivalent to p-invertibility of $\widehat{Q}^\circ$ and therefore of $\widehat{Q}$. $\square$

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