THE LIFTING PROBLEM FOR GALOIS REPRESENTATIONS

ALEXANDER MERKURJEV AND FEDERICO SCAVIA

Abstract. We solve the lifting problem for Galois representations in every dimension and in every characteristic. That is, we determine all pairs (n, k), where n is a positive integer and k is a field of characteristic $p > 0$, such that for every field F, every continuous homomorphism $\Gamma_F \to \mathrm{GL}_n(k)$ lifts to $\mathrm{GL}_n(W_2(k))$, where $W_2(k)$ is the ring of p-typical length 2 Witt vectors of k.

1. Introduction

1.1. **Lifting Galois representations.** Let F be a field, let Γ_F be the absolute Galois group of F, let k be a field of characteristic $p > 0$, let $W_2(k)$ be the ring of p-typical length 2 Witt vectors of k, and let n be a positive integer. Given an n-dimensional continuous k-linear representation V of Γ_F, a basic question is whether V lifts to $W_2(k)$, that is, whether there exists a continuous $W_2(k)$-free Γ_F-module W of rank n such that $W \otimes_{W_2(k)} k \cong V$. Similarly, for an n-dimensional complete flag of continuous Γ_F-representations, that is, a sequence of continuous Γ_F-representations $V_1 \subset V_2 \subset \cdots \subset V_n$ such that V_i has dimension i for all $1 \leq i \leq n$, one may ask whether the flag lifts to $W_2(k)$, that is, whether there exists a sequence of $W_2(k)$-free continuous $\Gamma_{\mathbb{F}}$-modules $W_1 \subset W_2 \subset \cdots \subset W_n$, such that W_{i+1}/W_i is $W_2(k)$-free for all $1 \leq i \leq n$, and which reduces to the sequence of the V_i after tensorization with k over $W_2(k)$.

The related question of existence of lifting representations of $\Gamma_{\mathbb{Q}}$ to characteristic zero, perhaps satisfying additional conditions, is of great importance in number theory. For example, given a continuous odd representation $\rho\colon \Gamma_{\mathbb{Q}} \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$, it is very useful to construct a continuous lifting $\tilde{\rho}\colon \Gamma_{\mathbb{Q}} \to \mathrm{GL}_2(\overline{\mathbb{Q}}_p)$ which is unramified outside finitely many places. The existence of such liftings, due to Ramakrishna [Ram99] and Khare–Wintenberger [KW09a], is a key tool in the proof Serre's modularity conjecture by Khare–Wintenberger [Kha06, KW09a, KW09b]. More generally, the deformation theory of continuous representations of absolute Galois groups of local and global fields is a prominent topic in number theory, with connections to modularity theorems and Wiles' proof of Fermat's Last Theorem [Wil95].

1.2. **The question of Khare and Serre.** Khare [Kha97] proved that, when k is a finite field, every 2-dimensional continuous representation of Γ_F with coefficients in k lifts to $W_2(k)$, for every field F. More precisely, Khare stated his theorem in the case when F is a number field, and Serre observed that Khare's argument generalized to an arbitrary field F; see [Kha97, Remark 2 p. 392]. Khare and Serre then asked whether every continuous finite-dimensional representation of Γ_F with coefficients in k lifts to $W_2(k)$; see [KL20, Question 1.1].

2020 *Mathematics Subject Classification.* 12G05; 11F80, 12F12, 20J06.

De Clercq and Florence [DCF22] generalized Khare's theorem by removing the assumption that k is finite (see Khare–Larsen [KL20] for an alternative proof) and showed that every continuous representation of Γ_F of dimension $n \leq 4$ over $\mathbb{F}_2$ lifts to $\mathbb{Z}/4\mathbb{Z}$. Florence [Flo20] conjectured that the question of Khare and Serre should have a positive answer, and even conjectured the stronger assertion that every finite-dimensional complete flag of continuous representations of Γ_F over k should lift to $W_2(k)$. He later constructed, for every odd prime p, a 3-dimensional complete flag of $\Gamma_{\mathbb{Q}((t))}$ which does not lift to $\mathbb{Z}/p^2\mathbb{Z}$, and amended his conjecture to include the assumption that F contains a primitive p^2-th root of unity; see [Flo24].

There are also positive results specific to local and global fields. By work of Böckle [Böc03], all continuous representations Γ_F-representations over $\mathbb{F}_p$ lift to $\mathbb{Z}/p^2\mathbb{Z}$, when F is a local field. The analogous statement for complete flags has recently been proved by Conti, Demarche and Florence [CDF24]. Böckle [Böc03] also proved lifting of certain mod p representations of Γ_F, when F is a global field. When F is a number field containing a primitive root of unity of order p^2, Khare and Larsen [KL20] proved that all 3-dimensional representations of Γ_F over $\mathbb{F}_p$ lift to $\mathbb{Z}/p^2\mathbb{Z}$.

1.3. **The main theorem.** In [MS24], we showed that the question of Khare and Serre and the conjecture of Florence have a negative answer, even over fields containing all p-primary roots of unity. More precisely, for all $n \geq 3$, all odd primes p, and all fields F containing a primitive p-th root of unity, letting $K \coloneqq F(x_1, \dots, x_p)$, where the x_i are independent variables over F, we constructed an n-dimensional continuous representation of Γ_K with $\mathbb{F}_p$ coefficients, admitting a Γ_K-invariant complete flag, and which does not lift to $\mathbb{Z}/p^2\mathbb{Z}$.

After this result, the goal shifted to determining all cases when the question of Khare and Serre has a positive answer, that is, the pairs (k, n), where k is a characteristic p field and n is a positive integer, such that, for every field K, every continuous n-dimensional representation of Γ_K lifts to $W_2(k)$. In this paper, we solve this problem. In fact, we answer a finer, relative version of the problem, where K ranges over all extensions of a fixed field F.

Theorem 1.1. *Let F be a field, let k be a field of characteristic $p > 0$, and let n be a positive integer. The following assertions are equivalent.*

(1) For every field extension K/F, every continuous n-dimensional representation of Γ_K over k lifts to $W_2(k)$.
(2) For every field extension K/F, every continuous n-dimensional complete flag of representations of Γ_K over k lifts to $W_2(k)$.
(3) At least one of the following conditions is satisfied:
  - *(a)* $\mathrm{char}(F) = p$,
  - *(b)* $n \leq 2$,
  - *(c)* $|k| = 2$ *and* $n \leq 4$.

Thus, for every pair (k, n) except for those considered by Khare and De Clercq–Florence, there exist n-dimensional Galois representations over k that do not lift to $W_2(k)$. In fact, in each such case, for every "generic" extension K/F, the Galois group Γ_K admits several "generic" non-liftable representations; see Section 6.7 for the precise statement.

1.4. **Negligible cohomology.** We rephrase Theorem 1.1 using the notion of negligible cohomology classes. Consider a short exact sequence of groups

$$0 \longrightarrow M \longrightarrow \tilde{G} \longrightarrow G \longrightarrow 1, \tag{1.1}$$

where M is abelian. The conjugation action of $\tilde{G}$ on M factors through G and makes M into a G-module; we let $\alpha \in H^2(G, M)$ be the class of (1.1). Following Serre, we say that α is *negligible over* F if for every field extension K/F and every continuous homomorphism $\rho\colon \Gamma_K \to M$ we have $\rho^*(\alpha) = 0$ in $H^2(K, M)$; see Section 2.1 for more details and references. Observe that α is negligible over F if and only if, for all field extensions K/F, every continuous homomorphism $\rho\colon \Gamma_K \to G$ lifts to a continuous homomorphism $\tilde{\rho}\colon \Gamma_K \to \tilde{G}$:

$$\begin{array}{ccccccccc} & & & & & & \Gamma_K & & \\ & & & & & \overset{\tilde{\rho}}{\swarrow} & \downarrow{\scriptstyle \rho} & & \\ 0 & \longrightarrow & M & \longrightarrow & \tilde{G} & \longrightarrow & G & \longrightarrow & 1. \end{array}$$

For every positive integer n and every field k of characteristic $p > 0$, we have the short exact sequences of groups

$$\mathrm{GLift}(k,n) \qquad 0 \longrightarrow M_n(k) \longrightarrow \mathrm{GL}_n(W_2(k)) \longrightarrow \mathrm{GL}_n(k) \longrightarrow 1,$$

$$\mathrm{BLift}(k,n) \qquad 0 \longrightarrow T_n(k) \longrightarrow B_n(W_2(k)) \longrightarrow B_n(k) \longrightarrow 1,$$

whose definition is recalled in Section 2.2. Here $B_n \subset \mathrm{GL}_n$ is the Borel subgroup of upper triangular matrices and $T_n(k) \subset M_n(k)$ is the subspace of upper triangular matrices. Since $W_2(\mathbb{F}_p) \cong \mathbb{Z}/p^2\mathbb{Z}$, for $k = \mathbb{F}_p$ these sequences take the form

$$\mathrm{GLift}(\mathbb{F}_p,n) \qquad 0 \longrightarrow M_n(\mathbb{F}_p) \longrightarrow \mathrm{GL}_n(\mathbb{Z}/p^2\mathbb{Z}) \longrightarrow \mathrm{GL}_n(\mathbb{F}_p) \longrightarrow 1,$$

$$\mathrm{BLift}(\mathbb{F}_p,n) \qquad 0 \longrightarrow T_n(\mathbb{F}_p) \longrightarrow B_n(\mathbb{Z}/p^2\mathbb{Z}) \longrightarrow B_n(\mathbb{F}_p) \longrightarrow 1,$$

where the maps $M_n(\mathbb{F}_p) \to \mathrm{GL}_n(\mathbb{Z}/p^2\mathbb{Z})$ and $T_n(\mathbb{F}_p) \to B_n(\mathbb{Z}/p^2\mathbb{Z})$ send the matrix A to $I + pA$.

A continuous n-dimensional representation of Γ_K over k may be lifted to $W_2(k)$ if and only if the corresponding continuous homomorphism $\Gamma_K \to \mathrm{GL}_n(k)$ (which is uniquely determined up to conjugation) lifts to a continuous homomorphism $\Gamma_K \to \mathrm{GL}_n(W_2(k))$. Similarly an n-dimensional complete flag of representations of Γ_K over k lifts to $W_2(k)$ if and only if the corresponding continuous homomorphism $\Gamma_K \to B_n(k)$ lifts to $B_n(W_2(k))$. Therefore Theorem 1.1 can be rephrased in the following equivalent way.

The classes of $\mathrm{GLift}(k,n)$ *and* $\mathrm{BLift}(k,n)$ *are negligible over* F *if* $\mathrm{char}(F) = p$, $n \leq 2$, *or* $|k| = 2$ *and* $n \leq 4$, *and are not negligible over* F *in all other cases.*

1.5. **Sketch of proof of the main theorem.** Our main tool for the proof of Theorem 1.1 is [MS24, Theorem 1.4] (see Theorem 2.2 below). Let $\alpha \in H^2(G, M)$ be the class of (1.1). Suppose that G is a finite group of exponent $e(G)$, that M has finite exponent $e(M)$, and that F contains a primitive root of unity of order $e(M)e(G)$. Under these assumptions, Theorem 2.2 asserts that α is negligible over F if and only if α belongs to the subgroup of $H^2(G, M)$ generated by all elements of the form $\mathrm{cor}_G^{G_a}(a \cup \chi)$, where $a \in M$, G_a is the stabilizer of a, and $\chi \in H^2(G_a, \mathbb{Z})$.

When k is finite, Theorem 2.2 reduces Theorem 1.1 to a problem in finite group cohomology, and when k is infinite, our strategy will be to apply Theorem 2.2 to suitable finite subgroups of $\mathrm{GL}_n(k)$.

We now sketch the proof of Theorem 1.1. For clarity, we only consider $\mathrm{GLift}(k,n)$. If $\mathrm{char}(F)=p$, then by [Ser02, Proposition 3 p. 75] the group $H^2(F,M)$ is trivial for every p-primary torsion Γ_F-module M, and so Theorem 1.1 is obvious in this case. When $\mathrm{char}(F)\neq p$, the theorems of Khare and De Clercq–Florence, of which we include self-contained proofs in Section 3, deal with all cases when $\mathrm{GLift}(k,n)$ has a positive solution. We must show that $\mathrm{GLift}(k,n)$ is not negligible over F in all the remaining cases.

For all $n\geq 3$ and all fields k of characteristic $p>0$, if the class of $\mathrm{GLift}(\mathbb{F}_p,n)$ is not negligible over F, neither is the class of $\mathrm{GLift}(k,n)$; see Lemma 4.1. Combining this with Theorem 2.5 (whose proof relies on Theorem 2.2) is enough to conclude when $n\geq 3$ and p is odd. It remains to consider the case when $p=2$ and $n\geq 3$. We consider the cases when $|k|>2$ and $|k|=2$ separately.

The case $p=2$, $|k|>2$ and $n\geq 3$ is handled in Section 5. By Lemma 2.8, it suffices to consider the case $n=3$. Since $|k|>2$, we may find a Klein subgroup W of k. Using W, we construct a Klein subgroup $Z\subset \mathrm{GL}_3(k)$ such that (i) the class of $\mathrm{GLift}(k,n)$ does not restrict to zero in $H^2(Z,M_3(k))$ (Lemma 5.1), while (ii) every class in $H^2(\mathrm{GL}_3(k),M_3(k))$ which is negligible over F is zero in $H^2(Z,M_3(k))$ (Lemma 5.2). Statement (i) is proved by a direct matrix computation, while (ii) crucially relies on Theorem 2.2. Because $\mathrm{GL}_3(k)$ is not necessarily finite, Theorem 2.2 does not apply to $\mathrm{GLift}(k,3)$; we get around this by considering a certain intermediate finite subgroup $Z\subset H\subset \mathrm{GL}_3(k)$ and by proving, using Theorem 2.2, the stronger statement that all classes in $H^2(H,M_3(k))$ that are negligible over F restrict to zero in $H^2(Z,M_3(k))$.

We consider the case when $|k|=2$ and $n\geq 5$ in Section 6. By Lemma 2.8, we may assume that $n=5$. Let $G:=\mathrm{GL}_5(\mathbb{F}_2)$. By Theorem 2.2, it suffices to prove that the class of $\mathrm{GLift}(\mathbb{F}_2,5)$ does not belong to the subgroup of $H^2(G,M_5(\mathbb{F}_2))$ generated by the elements $\mathrm{cor}_G^{G_A}(A\cup\chi)$, where A ranges over all elements of $M_5(\mathbb{F}_2)$, G_A is the stabilizer of A, and χ ranges over all elements of $H^2(G_A,\mathbb{Z})$. It suffices to consider a single $A\in M_5(\mathbb{F}_2)$ for each G-orbit. When A is not conjugate to a 5×5 Jordan block, a case-by-case analysis using the projection formula and matrix computations implies that $\mathrm{cor}_G^{G_A}(A\cup\chi)=0$ for all $\chi\in H^2(G_A,\mathbb{Z})$. When A is conjugate to a 5×5 Jordan block, we have $H^2(G_A,\mathbb{Z})=(\mathbb{Z}/2\mathbb{Z})\cdot\chi\oplus(\mathbb{Z}/8\mathbb{Z})\cdot\psi$ for some χ and ψ. Using the projection formula, we prove that $\mathrm{cor}_G^{G_A}(A\cup\chi)=0$. However, no such argument seems to be available for showing that $\mathrm{cor}_G^{G_A}(A\cup\psi)=0$. To overcome this, in Proposition 6.11 we construct a Klein subgroup $Z\subset\mathrm{GL}_5(\mathbb{F}_2)$ such that (i) the class of $\mathrm{GLift}(\mathbb{F}_2,5)$ restricts to a non-trivial element in $H^2(Z,M_5(\mathbb{F}_2))$ and (ii) the equality $\mathrm{res}_Z^G\,\mathrm{cor}_G^{G_A}(A\cup\psi)=0$ holds in $H^2(Z,M_5(\mathbb{F}_2))$; see Proposition 6.12. We prove (i) by a matrix computation, and (ii) by an intricate argument involving the double coset formula. This completes our proof sketch for Theorem 1.1.

For completeness, we also determine all cases when $\mathrm{GLift}(k,n)$ is split. In these cases, the corresponding lifting problem is trivial. As we prove in Theorem 7.1, the extension $\mathrm{GLift}(k,n)$ is split if and only if either $n=1$, or $n=2$ and $|k|\leq 3$, or $n=3$ and $|k|=2$.

Notation. For a commutative ring R and a non-negative integer n, we let $M_n(R)$ (resp. $T_n(R)$) be the R-algebra of $n\times n$ matrices (resp. upper triangular matrices) with coefficients in R. We also let $\mathrm{GL}_n(R)$ (resp. $B_n(R)$, resp. $U_n(R)$) be the group of invertible matrices (resp. upper triangular matrices, resp. upper unitriangular matrices) with coefficients in R, and we write $R^\times = \mathrm{GL}_1(R)$ for the group of units in R. For all $i, j \in \{1, \dots, n\}$, we let $E_{ij} \in M_n(R)$ be the matrix whose (i, j)-th entry is equal to 1 and whose other entries are equal to 0.

Let Γ be a profinite group. All group homomorphisms $\Gamma \to G$, where G is a group, will be assumed to be continuous for the profinite topology on Γ and the discrete topology on G. All Γ-modules will be assumed to be discrete. For every Γ-module M and every non-negative integer i, we let $H^i(\Gamma, M)$ be the i-th cohomology group.

If F is a field, we let Γ_F be the absolute Galois group of F and, for every Γ_F-module M and every $i \geq 0$, we let $H^i(F, M) \coloneqq H^i(\Gamma_F, M)$.

Let G be a group. For all $\sigma, \tau \in G$, we let $[\sigma, \tau] \coloneqq \sigma\tau\sigma^{-1}\tau^{-1}$ be the commutator of σ and τ. We let $[G, G]$ be the derived subgroup of G, and we let $G^{\mathrm{ab}} \coloneqq G/[G, G]$ be the abelianization of G.

Let M be a G-module. We often view M as a $\mathbb{Z}[G]$-module: for all $\sigma, \tau \in G$ and $m \in M$, we have $(\sigma + \tau)(m) = \sigma(m) + \tau(m)$ and $(\sigma\tau)(m) = \sigma(\tau(m))$. We write M^G for the subgroup of G-invariant elements of M. For a subgroup $H \subset G$, we let $\mathrm{res}^G_H \colon H^i(G, M) \to H^i(H, M)$ be the restriction map and, if H has finite index in G, we let $\mathrm{cor}^H_G \colon H^i(H, M) \to H^i(G, M)$ be the corestriction map. In degree 0, the corestriction $\mathrm{cor}^H_G \colon H^0(H, M) \to H^0(G, M)$ coincides with the norm map $N_{G/H} \colon M^H \to M^G$; see [NSW08, p. 48]. For every $\sigma \in G$ and every subgroup $H \subset G$, we let $\sigma_* \colon H^i(H, M) \to H^i(\sigma H \sigma^{-1}, M)$ be the conjugation map. By [NSW08, Proposition 1.5.6], for any two subgroups $H, K \subset G$ such that K has finite index in G, we have the double coset formula

$$\mathrm{res}^G_H \circ \mathrm{cor}^K_G = \sum_\sigma \mathrm{cor}^{H \cap \sigma K \sigma^{-1}}_H \circ \sigma_* \circ \mathrm{res}^K_{K \cap \sigma^{-1} H \sigma},$$

where σ ranges over a system of representatives of the double cosets $H \backslash G / K$.

Finally, for every $\sigma \in G$, we write M^σ for $M^{\langle \sigma \rangle}$ and N_σ for the norm map $N_{\langle \sigma \rangle / \{1\}} \colon M \to M^\sigma$, that is, the map given by $m \mapsto \sum_{i=0}^{e-1} \sigma^i m$, where e is the order of σ.

2. Preliminaries

2.1. **The lifting problem and negligible cohomology.** Let G be a group, let M be a G-module, let F be a field, and let $\alpha \in H^d(G, M)$ be a degree d cohomology class, for some $d \geq 0$. Following Serre, we say that α is *negligible over* F if for every field extension K/F and every homomorphism $\Gamma_K \to G$, the pullback map $H^d(G, M) \to H^d(K, M)$ takes α to zero; see [Ser91, Ser94] or [GMS03, §26 p. 61]. The negligible elements over F form a subgroup

$$H^2(G, M)_{\mathrm{neg}, F} \subset H^2(G, M).$$

Lemma 2.1. *Let F be a field, and let d be a non-negative integer.*

(1) *For every group G and every G-module homomorphism $M \to M'$, the induced map $H^d(G, M) \to H^d(G, M')$ takes the subgroup $H^d(G, M)_{\mathrm{neg}, F}$ into $H^d(G, M')_{\mathrm{neg}, F}$.*

(2) *For every group homomorphism $G' \to G$ and every G-module M, the pull-back map $H^d(G, M) \to H^d(G', M)$ takes the subgroup $H^d(G, M)_{\mathrm{neg},F}$ into $H^d(G', M)_{\mathrm{neg},F}$.*
(3) *For every field extension F'/F, every group G and every G-module M, we have $H^d(G, M)_{\mathrm{neg},F} \subset H^d(G, M)_{\mathrm{neg},F'}$.*
(4) *For every finite field extension F'/F, every group G and every G-module M, we have $[F' : F] \cdot H^d(G, M)_{\mathrm{neg},F'} \subset H^d(G, M)_{\mathrm{neg},F}$.*

Proof. The proofs immediately follow from the definitions; see [GM22, Proposition 2.3], where the assumption that G is finite is unnecessary. □

Let G be a group, let M be a G-module, and let F be a field. Consider a group extension

$$0 \longrightarrow M \longrightarrow \tilde{G} \longrightarrow G \longrightarrow 1, \tag{2.1}$$

where the G-action on M induced by the conjugation $\tilde{G}$-action coincides with the G-module action, and let $\alpha \in H^2(G, M)$ be the class of (2.1). The class α is negligible if and only if, for every field extension K/F, every homomorphism $\Gamma_K \to G$ lifts to a homomorphism $\Gamma_K \to \tilde{G}$.

In [MS24], we determined the subgroup $H^2(G, M)_{\mathrm{neg},F} \subset H^2(G, M)$ when G is finite, M has finite exponent, and F contains enough roots of unity.

Theorem 2.2. *Let G be a finite group of exponent $e(G)$, let M be a G-module of finite exponent $e(M)$, and let F be a field containing a primitive root of unity of order $e(M)e(G)$. Then $H^2(G, M)_{\mathrm{neg},F}$ is generated by all elements of the form $\mathrm{cor}^H_G(a \cup \chi)$, where H is a subgroup of G, $a \in M^H$ and $\chi \in H^2(H, \mathbb{Z})$.*

In fact, $H^2(G, M)_{\mathrm{neg},F}$ is generated by all elements of the form $\mathrm{cor}^{G_a}_G(a \cup \chi)$, where a ranges over all elements of M, G_a is the stabilizer of a in G, and χ ranges over all elements of $H^2(G_a, \mathbb{Z})$.

Proof. When M is finite, this is [MS24, Theorem 1.3]. The general case follows from the finite case by writing M as the union of its finite G-submodules. □

Suppose that the group G is finite. For every subgroup $H \subset G$, we define

$$\varphi_H \colon M^H \otimes H^2(H, \mathbb{Z}) \xrightarrow{\cup} H^2(H, M) \xrightarrow{\mathrm{cor}} H^2(G, M). \tag{2.2}$$

Therefore, under the assumptions of Theorem 2.2, the subgroup $H^2(G, M)_{\mathrm{neg},F}$ is generated by the images of φ_{G_a}, where a ranges over all elements of M, and where G_a is the stabilizer of a in G. In fact, as the next lemma shows, it suffices to consider a single $a \in M$ for each G-orbit.

Lemma 2.3. *Let G be a finite group, let M be a G-module. For every $g \in G$ and every $a \in M$, we have $\mathrm{Im}(\varphi_{ga}) = \mathrm{Im}(\varphi_a)$.*

Proof. Let $a \in M$, let $g \in G$, and set $a' := ga$. We have a commutative diagram

$$\begin{array}{ccccc}
M^{G_a} \otimes H^2(G_a, \mathbb{Z}) & \xrightarrow{\cup} & H^2(G_a, M) & \xrightarrow{\mathrm{cor}} & H^2(G, M) \\
\downarrow{\scriptstyle g_* \otimes g_*} & & \downarrow{\scriptstyle g_*} & & \downarrow{\scriptstyle g_*} \\
M^{G_{a'}} \otimes H^2(G_{a'}, \mathbb{Z}) & \xrightarrow{\cup} & H^2(G_{a'}, M) & \xrightarrow{\mathrm{cor}} & H^2(G, M).
\end{array}$$

Here, the left square commutes by [NSW08, Proposition 1.5.3(i)], and the right square commutes by [NSW08, Proposition 1.5.4]. Moreover, by [Wei94, Theorem 6.7.8] the right vertical map is the identity. It follows that $\mathrm{Im}(\varphi_{G_a}) = \mathrm{Im}(\varphi_{G_{a'}})$. □

2.2. **Length 2 Witt vectors.** Let k be a field of characteristic $p > 0$. We recall the definition of the p-typical length 2 Witt vectors $W_2(k)$ of k. Consider the polynomial $\Phi(x,y) \coloneqq ((x+y)^p - x^p - y^p)/p \in \mathbb{Z}[x,y]$. As a set $W_2(k) \coloneqq k \times k$, and, for all (a_1,b_1), $(a_2,b_2) \in W_2(k)$, one has

$$(a_1,b_1) + (a_2,b_2) \coloneqq (a_1 + a_2, b_1 + b_2 - \Phi(a_1,a_2)),$$
$$(a_1,b_1) \cdot (a_2,b_2) \coloneqq (a_1a_2, a_1^p b_2 + a_2^p b_1);$$

see [Bou06, Chapitre IX, §1, paragraphe 4]. We have a short exact sequence of abelian groups

$$0 \longrightarrow k \xrightarrow{\iota} W_2(k) \xrightarrow{\pi} k \longrightarrow 0, \tag{2.3}$$

where $\pi(a,b) = a$ and $\iota(b) = (0,b)$ for all $a,b \in k$. The map π is a ring homomorphism. For every integer $n \geq 0$, we obtain a short exact sequence of groups

$$\mathrm{Lift}(k,n) \qquad 0 \longrightarrow M_n(k) \longrightarrow \mathrm{GL}_n(W_2(k)) \longrightarrow \mathrm{GL}_n(k) \longrightarrow 1,$$

where the homomorphism $\mathrm{GL}_n(W_2(k)) \to \mathrm{GL}_n(k)$ is induced by π, and where the inclusion $M_n(k) \to \mathrm{GL}_n(W_2(k))$ is given by $(m_{ij}) \mapsto I + (\iota(m_{ij}))$. Similarly, we have an exact sequence

$$\mathrm{BLift}(k,n) \qquad 0 \longrightarrow T_n(k) \longrightarrow B_n(W_2(k)) \longrightarrow B_n(k) \longrightarrow 1.$$

For every $A = (a_{ij}) \in \mathrm{GL}_n(k)$, we define $A^{(p)} \coloneqq (a_{ij}^p) \in \mathrm{GL}_n(k)$.

Lemma 2.4. *The* $\mathrm{GL}_n(k)$*-action on* $M_n(k)$ *induced by* $\mathrm{GLift}(k,n)$ *is given by*

$$\mathrm{GL}_n(k) \times M_n(k) \to M_n(k), \qquad (A,M) \mapsto A^{(p)}M(A^{(p)})^{-1}.$$

Proof. Under the identification $M_n(W_2(k)) = M_n(k) \times M_n(k)$ induced by the identification $W_2(k) = k \times k$, the conclusion amounts to

$$(A,0)(0,M)(A,0)^{-1} = (0, A^{(p)}M(A^{(p)})^{-1})$$

for all $A \in \mathrm{GL}_n(k)$ and $M \in M_n(k)$. This is equivalent to

$$(A,0)(0,M) = (0, A^{(p)}M(A^{(p)})^{-1})(A,0),$$

which follows from the identities

$$(X,0)(0,Y) = (0, X^{(p)}Y), \qquad (0,Y)(X,0) = (0, YX^{(p)}),$$

valid for all $X, Y \in M_n(k)$. □

When $k = \mathbb{F}_p$, we have a ring isomorphism $\mathbb{Z}/p^2\mathbb{Z} \xrightarrow{\sim} W_2(\mathbb{F}_p)$ determined by $1 + p^2\mathbb{Z} \mapsto (1,0)$. Thus (2.3) becomes

$$0 \longrightarrow \mathbb{F}_p \longrightarrow \mathbb{Z}/p^2\mathbb{Z} \longrightarrow \mathbb{F}_p \longrightarrow 0,$$

where the map $\mathbb{F}_p \to \mathbb{Z}/p^2\mathbb{Z}$ sends 1 to $p + p^2\mathbb{Z}$, and the sequences $\mathrm{GLift}(\mathbb{F}_p,n)$ and $\mathrm{BLift}(\mathbb{F}_p,n)$ take the form

$$\mathrm{GLift}(\mathbb{F}_p,n) \qquad 0 \longrightarrow M_n(\mathbb{F}_p) \longrightarrow \mathrm{GL}_n(\mathbb{Z}/p^2\mathbb{Z}) \longrightarrow \mathrm{GL}_n(\mathbb{F}_p) \longrightarrow 1,$$

$$\mathrm{BLift}(\mathbb{F}_p, n) \qquad 0 \longrightarrow T_n(\mathbb{F}_p) \longrightarrow B_n(\mathbb{Z}/p^2\mathbb{Z}) \longrightarrow B_n(\mathbb{F}_p) \longrightarrow 1,$$

where the maps $M_n(\mathbb{F}_p) \to \mathrm{GL}_n(\mathbb{Z}/p^2\mathbb{Z})$ and $T_n(\mathbb{F}_p) \to B_n(\mathbb{Z}/p^2\mathbb{Z})$ send A to $I + pA$. By Lemma 2.4, the induced $\mathrm{GL}_n(\mathbb{F}_p)$-action on $M_n(\mathbb{F}_p)$ is given by matrix conjugation.

Theorem 2.5. *For all $n \geq 3$, all odd primes p, and all fields F of characteristic different from p, the class of $\mathrm{GLift}(\mathbb{F}_p, n)$ is not negligible over F.*

Proof. See [MS24, Theorem 5.1]. □

We conclude this subsection with some basic observations about $\mathrm{GLift}(k, n)$ and $\mathrm{BLift}(k, n)$.

Lemma 2.6. *Let Γ be a profinite group, let k be a field of characteristic $p > 0$, and let V be a finite-dimensional k-representation of Γ. There exists an open subgroup $\Gamma' \subset \Gamma$ of prime-to-p index such that V is a unitriangular representation of Γ'.*

Proof. Replacing Γ with the image of the natural homomorphism $\Gamma \to \mathrm{Aut}(V)$, we may assume that Γ is finite. Replacing Γ by a p-Sylow subgroup, we may assume that Γ is a p-group. By induction on the dimension of V, it suffices to show that $V^\Gamma \neq \{0\}$. This is proved in [Ser12, Proposition 26 p. 64]. □

Lemma 2.7. *Let F be a field, let k be a field of characteristic $p > 0$, and let n be a positive integer. If the class of $\mathrm{BLift}(k, n)$ is negligible over F, then so is the class of $\mathrm{GLift}(k, n)$.*

Proof. Let K/F be a field extension, and let $\rho\colon \Gamma_K \to \mathrm{GL}_n(k)$ be a group homomorphism. Let $G \subset \mathrm{GL}_n(k)$ be the image of ρ, and let $P \subset G$ be a p-Sylow subgroup of G. Since P is a finite p-group, by Lemma 2.6, we may assume that $P \subset B_n(k)$. Let $c \in H^2(B_n(k), M_n(k))$ be the pushforward of the class of $\mathrm{BLift}(k, n)$. By Lemma 2.1(1), the class c is negligible over F, and hence so is its restriction in $H^2(P, M_n(k))$. The latter class is the restriction of the class of $\mathrm{GLift}(k, n)$ via the inclusion $P \hookrightarrow G \hookrightarrow \mathrm{GL}_n(k)$. As $[G : P]$ is prime to p, by Lemma 2.1(4) the restriction in $H^2(G, M_n(k))$ of the class of $\mathrm{GLift}(k, n)$ is also negligible over F. It follows that ρ lifts to $\mathrm{GL}_n(W_2(k))$. □

Lemma 2.8. *Let F be a field, let k be a field of characteristic $p > 0$, and let $n \geq m$ be positive integers. If the class of $\mathrm{GLift}(k, n)$ is negligible over F, then so is the class of $\mathrm{GLift}(k, m)$. Similarly, if the class of $\mathrm{BLift}(k, n)$ is negligible over F, then so is the class of $\mathrm{BLift}(k, m)$.*

Proof. See [DCF22, Lemma 3.4]. For a more direct argument, see [MS24, Lemma 5.3], which is stated and proved only when $k = \mathbb{F}_p$, but whose proof immediately generalizes to arbitrary k. □

2.3. **Extensions of bicyclic groups.** Let s and t be positive integers, let

$$Z \coloneqq \langle \rho, \mu \mid \rho^s = \mu^t = [\rho, \mu] = 1 \rangle$$

be a bicyclic group of order st, and let M be a Z-module. Define the abelian group

$$Z^2(Z, M) \coloneqq \{(a, b, c) \in M^3 \mid \rho(a) = a, \mu(b) = b, N_\rho(c) = (\mu - 1)a, N_\mu(c) = (\rho - 1)b\},$$

its subgroup

$$B^2(Z, M) = \{(N_\rho(u), N_\mu(v), (\rho - 1)v + (\mu - 1)u) \mid u, v \in M\},$$

and the quotient group

$$\tilde{H}^2(Z,M) \coloneqq Z^2(Z,M)/B^2(Z,M).$$

Let $\alpha \in H^2(Z,M)$, and let

$$0 \longrightarrow M \longrightarrow \tilde{Z} \longrightarrow Z \longrightarrow 1. \tag{2.4}$$

be a group extension representing α. Let $\tilde{\rho}, \tilde{\mu} \in \tilde{Z}$ be lifts of ρ and μ, respectively. Observe that $\tilde{\rho}^{-s}$, $\tilde{\mu}^t$ and $[\tilde{\rho}, \tilde{\mu}]$ belong to M, and the triple $(\tilde{\rho}^{-s}, \tilde{\mu}^t, [\tilde{\rho}, \tilde{\mu}])$ belongs to $Z^2(Z,M)$. Define

$$f(\alpha) \coloneqq (\tilde{\rho}^{-s}, \tilde{\mu}^t, [\tilde{\rho}, \tilde{\mu}]) + B^2(Z,M) \in \tilde{H}^2(Z,M).$$

Lemma 2.9. *This construction yields a well-defined function*

$$f\colon H^2(Z,M) \to \tilde{H}^2(Z,M)$$

such that $f(0) = 0$.

Proof. Let $\alpha \in H^2(Z,M)$, let (2.4) be a group extension representing α, and let $\tilde{\rho}, \tilde{\mu} \in \tilde{Z}$ be lifts of ρ and μ, respectively.

We first show that $f(\alpha)$ does not depend on the choice of lifts $\tilde{\rho}, \tilde{\mu}$. Any other pair of lifts has the form $u^{-1}\tilde{\rho}, v\tilde{\mu}$ for some $u, v \in M$. (Here we view M as a subgroup of $\tilde{Z}$, and hence use multiplicative notation for the group operation.) Then

$$(u^{-1}\tilde{\rho})^{-s} = \tilde{\rho}^{-s} N_\rho(u), \qquad (v\tilde{\mu})^t = N_\mu(v)\tilde{\mu}^t,$$
$$[u^{-1}\tilde{\rho}, v\tilde{\mu}] = u^{-1}\tilde{\rho}v\tilde{\rho}^{-1}\tilde{\rho}\tilde{\mu}\tilde{\rho}^{-1}\tilde{\mu}^{-1}\tilde{\mu}u\tilde{\mu}^{-1}v^{-1} = u^{-1}\rho(v)[\tilde{\rho}, \tilde{\mu}]\mu(u)v^{-1}.$$

Recalling that the subgroup M is abelian, we obtain, in additive notation,

$$((u^{-1}\tilde{\rho})^{-s}, (v\tilde{\mu})^t, [u^{-1}\tilde{\rho}, v\tilde{\mu}]) = (\tilde{\rho}^{-s}, \tilde{\mu}^t, [\tilde{\rho}, \tilde{\mu}]) + (N_\rho(u), N_\mu(v), (\rho-1)v + (\mu-1)u)$$

in $Z^2(Z,M)$. Thus $f(\alpha)$ does not depend on the choice of lift. The fact that $f(\alpha)$ does not depend on the choice of the group extension (2.4) is clear. Finally, if $\alpha = 0$ then (2.4) admits a splitting $s\colon Z \to \tilde{Z}$. Letting $\tilde{\rho} \coloneqq s(\rho)$ and $\tilde{\mu} \coloneqq s(\mu)$, we have $(\tilde{\rho}^{-s}, \tilde{\mu}^t, [\tilde{\rho}, \tilde{\mu}]) = 0$ in $Z^2(Z,M)$, and hence $f(0) = 0$. ☐

Remark 2.10. One can show that the function $f\colon H^2(Z,M) \to \tilde{H}^2(Z,M)$ is a group isomorphism. We will not need this stronger assertion.

3. Proofs of the theorems of Khare and De Clercq–Florence

Let R be a commutative ring, let Γ be a profinite group, and let A and C be $R[\Gamma]$-modules. We let

$$\mathrm{Ext}^1_{R[\Gamma],s}(C,A) \coloneqq \mathrm{Ker}[\mathrm{Ext}^1_{R[\Gamma]}(C,A) \to \mathrm{Ext}^1_R(C,A)]$$

be the abelian group of isomorphism classes of R-split exact sequences of $R[\Gamma]$-modules

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0.$$

Given a (continuous) 1-cocycle φ of Γ with values in $\mathrm{Hom}_R(C,A)$, one introduces the structure of an $R[\Gamma]$-module on $A \oplus C$ by the formula

$$g(a,c) = (ga + \varphi(g)(gc), gc).$$

This yields a group isomorphism

$$H^1(\Gamma, \mathrm{Hom}_R(C,A)) \xrightarrow{\sim} \mathrm{Ext}^1_{R[\Gamma],s}(C,A). \tag{3.1}$$

For every ring homomorphism $R \to R'$, letting $A' := A \otimes_R R'$ and $C' := C \otimes_R R'$, base change induces a commutative square

$$\begin{array}{ccc} H^1(\Gamma, \mathrm{Hom}_R(C,A)) & \xrightarrow{\sim} & \mathrm{Ext}^1_{R[\Gamma],s}(C,A) \\ \downarrow & & \downarrow \\ H^1(\Gamma, \mathrm{Hom}_{R'}(C',A')) & \xrightarrow{\sim} & \mathrm{Ext}^1_{R'[\Gamma],s}(C',A'), \end{array} \tag{3.2}$$

where the bottom horizontal map is (3.1) for the R'-modules A' and C'.

The following theorem was proved by Khare [Kha97] when the field k is finite, and by De Clercq and Florence [DCF22] in general.

Theorem 3.1. *For every field F and every field k of characteristic $p > 0$, the classes of* $\mathrm{GLift}(k, 2)$ *and* $\mathrm{BLift}(k, 2)$ *are negligible over F.*

Proof. By Lemma 2.7, it suffices to prove that the class of $\mathrm{BLift}(k, 2)$ is negligible over F. By [Ser02, Proposition 3 p. 75], we may assume that $\mathrm{char}(F) \neq p$ and, by Lemma 2.1(4), that F contains a primitive p-th root of unity ζ. The choice of ζ allows us to identify μ_p with $\mathbb{Z}/p\mathbb{Z}$ and $k \otimes \mu_p$ with k. Let

$$0 \longrightarrow k \longrightarrow V \longrightarrow k \longrightarrow 0 \tag{3.3}$$

be a 2-dimensional complete flag of representations of Γ_F over k. By Lemma 2.6, there exists an open subgroup $\Gamma' \subset \Gamma$ acting trivially on both copies of k in (3.3). By Lemma 2.1(4), we may replace Γ by Γ', that is, we may assume that Γ_F acts trivially on both copies of k in (3.3). Consider the commutative diagram

$$\begin{array}{ccccc} W_2(k) \otimes F^\times & \xrightarrow{\sim} & W_2(k) \otimes H^1(F, \mu_{p^2}) & \xrightarrow{\cup} & H^1(F, W_2(k) \otimes \mu_{p^2}) \\ \downarrow & & \downarrow & & \downarrow \\ k \otimes F^\times & \xrightarrow{\sim} & k \otimes H^1(F, \mu_p) & \xrightarrow{\cup} & H^1(F, k), \end{array}$$

where the Γ_F-action on k and $W_2(k)$ is trivial, the vertical maps are induced by the reduction map $W_2(k) \to k$, and the left horizontal maps are induced by the Kummer sequence. As the map $W_2(k) \to k$ is surjective, so is $W_2(k) \otimes F^\times \to k \otimes F^\times$. Since k is an $\mathbb{F}_p$-vector space, the bottom-right map is an isomorphism, and hence the homomorphism $H^1(F, W_2(k) \otimes \mu_{p^2}) \to H^1(F, k)$ is also surjective.

In view of (3.1), the extension (3.3) is represented by a class $\alpha \in H^1(F, k)$ and, letting $\tilde{\alpha} \in H^1(F, W_2(k) \otimes \mu_{p^2})$ be a lift of α, the class $\tilde{\alpha}$ represents a $W_2(k)$-split extension of $W_2(k)[\Gamma_F]$-modules

$$0 \longrightarrow W_2(k) \otimes \mu_{p^2} \longrightarrow W \longrightarrow W_2(k) \longrightarrow 0. \tag{3.4}$$

Since $\tilde{\alpha}$ lifts α, the commutativity of (3.2) (where the homomorphism $R \to R'$ is the reduction map $W_2(k) \to k$) implies that tensoring (3.4) with k over $W_2(k)$ yields (3.3), and the conclusion follows. □

The next theorem is due to De Clercq and Florence [DCF22, Corollary 6.3].

Theorem 3.2. *For every field F and every $n \leq 4$, the classes of* $\mathrm{GLift}(\mathbb{F}_2, n)$ *and* $\mathrm{BLift}(\mathbb{F}_2, n)$ *are negligible over F.*

Proof. By Lemma 2.7, it suffices to prove that $\mathrm{BLift}(\mathbb{F}_2, n)$ is negligible over F. By [Ser02, Proposition 3 p. 75], we may assume that $\mathrm{char}(F) \neq 2$ and, by Lemma 2.8, that $n = 4$.

Let $V_1 \subset V_2 \subset V_3 \subset V_4 = V$ be a 4-dimensional complete flag of representations of Γ_F over $\mathbb{F}_2$. Every triangular action of Γ_F on a 2-dimensional vector space over $\mathbb{F}_2$ has a permutation basis: this is clear if the Γ_F-action is trivial, and if the Γ_F-action is non-trivial, then the Γ_F-orbit of a non-fixed vector is a permutation basis. Thus, there exist Γ_F-invariant bases $X = \{x_1, x_2\}$ of V_2 and $Y = \{y_1, y_2\}$ of V/V_2 such that

$$V_2 = \mathbb{F}_2[X], \quad V/V_2 = \mathbb{F}_2[Y], \quad V_1 = \mathbb{F}_2 \cdot (x_1 + x_2), \quad V_3/V_2 = \mathbb{F}_2 \cdot (y_1 + y_2).$$

We obtain a short exact sequence of $\mathbb{F}_2$-linear Γ_F-representations

$$0 \longrightarrow \mathbb{F}_2[X] \longrightarrow V \longrightarrow \mathbb{F}_2[Y] \longrightarrow 0. \tag{3.5}$$

Let L be an étale F-algebra corresponding to the Γ_F-set $X \times Y$; see [KMRT98, Theorem 18.4]. We have a commutative square of $(\mathbb{Z}/4\mathbb{Z})[\Gamma_F]$-modules

$$\begin{array}{ccc} \mathrm{Hom}_{\mathbb{Z}/4\mathbb{Z}}((\mathbb{Z}/4\mathbb{Z})[Y], \mu_4[X]) & \xrightarrow{\sim} & \mu_4[X \times Y] \\ \downarrow & & \downarrow \\ \mathrm{Hom}_{\mathbb{F}_2}(\mathbb{F}_2[Y], \mathbb{F}_2[X]) & \xrightarrow{\sim} & \mathbb{F}_2[X \times Y], \end{array}$$

where $\mu_4[X] \coloneqq (\mathbb{Z}/4\mathbb{Z})[X] \otimes \mu_4$ and $\mu_4[X \times Y] \coloneqq (\mathbb{Z}/4\mathbb{Z})[X \times Y] \otimes \mu_4$. We obtain a commutative diagram

$$\begin{array}{ccccc} H^1(L, \mu_4) & \xrightarrow{\sim} & H^1(F, \mu_4[X \times Y]) & \xrightarrow{\sim} & H^1(F, \mathrm{Hom}_{\mathbb{Z}/4\mathbb{Z}}((\mathbb{Z}/4\mathbb{Z})[Y], \mu_4[X])) \\ \downarrow & & \downarrow & & \downarrow \\ H^1(L, \mathbb{F}_2) & \xrightarrow{\sim} & H^1(F, \mathbb{F}_2[X \times Y]) & \xrightarrow{\sim} & H^1(F, \mathrm{Hom}_{\mathbb{F}_2}(\mathbb{F}_2[Y], \mathbb{F}_2[X])), \end{array}$$

where the three vertical arrows are induced by the reduction maps $\mathbb{Z}/4\mathbb{Z} \to \mathbb{F}_2$ and $\mu_4 \to \mathbb{F}_2$, and where the left horizontal maps are the Faddeev–Shapiro isomorphisms; see [NSW08, Proposition 1.6.4]. The map $H^1(L, \mu_4) \to H^1(L, \mathbb{F}_2)$ is surjective by Kummer theory, and hence all vertical maps are surjective.

Let $\alpha \in H^1(F, \mathrm{Hom}_{\mathbb{F}_2}(\mathbb{F}_2[Y], \mathbb{F}_2[X]))$ be the class of (3.5), and lift α to an element $\tilde{\alpha} \in H^1(F, \mathrm{Hom}_{\mathbb{Z}/4\mathbb{Z}}((\mathbb{Z}/4\mathbb{Z})[Y], \mu_4[X]))$. Then, under the identification of (3.1), $\tilde{\alpha}$ is the class of a $(\mathbb{Z}/4\mathbb{Z})$-split exact sequence of $(\mathbb{Z}/4\mathbb{Z})[\Gamma_F]$-modules

$$0 \longrightarrow \mu_4[X] \longrightarrow W \longrightarrow (\mathbb{Z}/4\mathbb{Z})[Y] \longrightarrow 0$$

which reduces to (3.5) modulo 2. Define $W_1 \coloneqq \mu_4 \cdot (x_1 + x_2)$, $W_2 \coloneqq \mu_4[X]$, let W_3 be the inverse image of $\mathbb{Z}/4\mathbb{Z} \cdot (y_1 + y_2)$ in W, and let $W_4 \coloneqq W$. Then the flag of $\mathbb{Z}/4\mathbb{Z}$-free Γ_F-modules $W_1 \subset W_2 \subset W_3 \subset W_4$ reduces to the flag $V_1 \subset V_2 \subset V_3 \subset V_4$ modulo 2, as desired. □

4. Proof of Theorem 1.1 for odd p and $n \geq 3$

Lemma 4.1. *Let p be a prime number, let n be a positive integer, let F be a field, and let k be a field of characteristic p. If $\mathrm{GLift}(\mathbb{F}_p, n)$ is not negligible over F, neither is $\mathrm{GLift}(k, n)$. Similarly, if $\mathrm{BLift}(\mathbb{F}_p, n)$ is not negligible over F, neither is $\mathrm{BLift}(k, n)$.*

Proof. We have a commutative diagram of abelian groups with exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & k & \longrightarrow & W_2(k) & \longrightarrow & k & \longrightarrow & 0 \\
 & & \| & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & k & \longrightarrow & C & \longrightarrow & \mathbb{F}_p & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \lambda} & & \downarrow{\scriptstyle \varphi} & & \| & & \\
0 & \longrightarrow & \mathbb{F}_p & \longrightarrow & W_2(\mathbb{F}_p) & \longrightarrow & \mathbb{F}_p & \longrightarrow & 0,
\end{array}$$

where the group homomorphism λ is a splitting of the inclusion $\mathbb{F}_p \hookrightarrow k$. By definition, C is the subring of $W_2(k)$ consisting of those pairs (a,b) such that $a \in \mathbb{F}_p$ and $b \in k$. The ring homomorphism φ is given by $\varphi(a,b) = (a, \lambda(b))$.

We obtain a commutative diagram of groups with exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & M_n(k) & \longrightarrow & \mathrm{GL}_n(W_2(k)) & \longrightarrow & \mathrm{GL}_n(k) & \longrightarrow & 1 \\
 & & \| & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & M_n(k) & \longrightarrow & \mathrm{GL}_n(C) & \longrightarrow & \mathrm{GL}_n(\mathbb{F}_p) & \longrightarrow & 1 \\
 & & \downarrow{\scriptstyle \lambda_*} & & \downarrow{\scriptstyle \varphi_*} & & \| & & \\
0 & \longrightarrow & M_n(\mathbb{F}_p) & \longrightarrow & \mathrm{GL}_n(W_2(\mathbb{F}_p)) & \longrightarrow & \mathrm{GL}_n(\mathbb{F}_p) & \longrightarrow & 1.
\end{array}$$

Since the top row is negligible over F, by Lemma 2.1(2) so is the middle row, and hence by Lemma 2.1(1) so is the bottom row. The proof for $\mathrm{BLift}(k,n)$ is entirely analogous. □

Proof of Theorem 1.1 for odd p and $n \geq 3$. By Lemma 2.7, it suffices to prove that $\mathrm{GLift}(k,n)$ is not negligible over F for all $n \geq 3$. By Lemma 4.1, it is enough to show that the class of $\mathrm{GLift}(\mathbb{F}_p, n)$ is not negligible over F, which follows from Theorem 2.5. □

5. Proof of Theorem 1.1 for $p = 2$, $|k| > 2$ and $n \geq 3$

Lemma 5.1. *Let $n \geq 2$ be an integer, let k be a field of characteristic 2 such that $|k| > 2$, let $x, y \in k^\times$ be two distinct elements, and let $Z \subset U_n(k)$ be the Klein subgroup generated by $\rho \coloneqq I + xE_{1,n}$ and $\mu \coloneqq I + yE_{1,n}$. The class of $\mathrm{GLift}(k,n)$ in $H^2(\mathrm{GL}_n(k), M_n(k))$ restricts to a non-trivial class in $H^2(Z, M_n(k))$.*

Proof. We first reduce to the case when $n = 2$. We have a commutative diagram with exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & M_n(k) & \longrightarrow & \mathrm{GL}_n(W_2(k)) & \longrightarrow & \mathrm{GL}_n(k) & \longrightarrow & 1 \\
 & & \| & & \uparrow & & \uparrow{\scriptstyle \iota} & & \\
0 & \longrightarrow & M_n(k) & \longrightarrow & E & \longrightarrow & \mathrm{GL}_2(k) & \longrightarrow & 1 \\
 & & \downarrow{\scriptstyle \pi} & & \downarrow & & \| & & \\
0 & \longrightarrow & M_2(k) & \longrightarrow & \mathrm{GL}_2(W_2(k)) & \longrightarrow & \mathrm{GL}_2(k) & \longrightarrow & 1,
\end{array}$$

where π and ι are given by

$$[a_{ij}] \mapsto \begin{bmatrix} a_{1,1} & a_{1,n} \\ a_{n,1} & a_{n,n} \end{bmatrix}, \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} a & 0 & \cdots & 0 & b \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \\ c & 0 & \cdots & 0 & d \end{pmatrix},$$

respectively. Letting $\alpha_n \in H^2(\mathrm{GL}_n(k), M_n(k))$ be the class of $\mathrm{GLift}(k,n)$, we deduce that $\alpha_2 = \pi_*\iota^*(\alpha_n)$. Let $j_n\colon Z \hookrightarrow \mathrm{GL}_n(k)$ be the inclusion map. We have $\iota \circ j_2 = j_n$, so that $j_n^* = j_2^*\iota^*$, and we have $j_2^*\pi_* = \pi_* j_2^*$. Thus

$$j_2^*(\alpha_2) = j_2^*\pi_*\iota^*(\alpha_n) = \pi_* j_2^*\iota^*(\alpha_n) = \pi_* j_n^*(\alpha_n)$$

in $H^2(Z, M_2(k))$. In particular, if $j_2^*(\alpha_2) \neq 0$ in $H^2(Z, M_2(k))$, then $j_n^*(\alpha_n) \neq 0$ in $H^2(Z, M_n(k))$. We may thus assume that $n = 2$.

Let $\tilde{x} \coloneqq (x, 0)$ and $\tilde{y} \coloneqq (y, 0)$ be lifts of x and y in $W_2(k)$, respectively, and define $\tilde{\rho} \coloneqq I + \tilde{x}E_{12}$ and $\tilde{\mu} \coloneqq I + \tilde{y}E_{12}$ in $\mathrm{GL}_2(W_2(k))$. Then $\tilde{\rho}$ and $\tilde{\mu}$ lift ρ and μ, respectively. For every $u \in k$, we have $(u, 0) + (u, 0) = (0, u^2) = \iota(u^2)$ in $W_2(k)$, where the map $\iota\colon k \to W_2(k)$ has been defined in (2.3). Thus

$$\tilde{\rho}^{-2} = I - \iota(x^2)E_{12} = I + \iota(x^2)E_{12}, \qquad \tilde{\mu}^2 = I + \iota(y^2)E_{12}, \qquad [\tilde{\rho}, \tilde{\mu}] = I$$

in $\mathrm{GL}_2(W_2(k))$.

Suppose by contradiction that $\mathrm{GLift}(k, 2)$ is trivial. Then, by Lemma 2.9, there exist U and V in $M_2(k)$ such that

$$N_\rho(U) = x^2E_{12}, \qquad N_\mu(V) = y^2E_{12}, \qquad N_\mu(U) = N_\rho(V)$$

in $M_2(k)$, that is, letting $U = (u_{ij})$ and $V = (v_{ij})$,

$$\begin{bmatrix} x^2u_{21} & x^2u_{11} + x^4u_{21} + x^2u_{22} \\ 0 & x^2u_{21} \end{bmatrix} = \begin{bmatrix} 0 & x^2 \\ 0 & 0 \end{bmatrix},$$

$$\begin{bmatrix} y^2v_{21} & y^2v_{11} + y^4v_{21} + y^2v_{22} \\ 0 & y^2v_{21} \end{bmatrix} = \begin{bmatrix} 0 & y^2 \\ 0 & 0 \end{bmatrix},$$

$$\begin{bmatrix} y^2u_{21} & y^2u_{11} + y^4u_{21} + y^2u_{22} \\ 0 & y^2u_{21} \end{bmatrix} = \begin{bmatrix} x^2v_{21} & x^2v_{11} + x^4v_{21} + x^2v_{22} \\ 0 & x^2v_{21} \end{bmatrix}.$$

It remains to show that no such U and V exist. Indeed, if they existed, then

$$u_{21} = 0 = v_{21},$$

$$u_{11} + u_{22} = 1 = v_{11} + v_{22},$$
$$y^2u_{11} + y^2u_{22} = x^2v_{11} + x^2v_{22},$$

which would imply $x^2 = y^2$ and hence $x = y$, a contradiction. □

Lemma 5.2. *Let F be a field of characteristic different from 2, let k be a field of characteristic 2 such that $|k| > 2$, let $W \subset k$ be a finite subgroup such that $|W| > 2$, let $H \subset U_3(k)$ be the finite subgroup*

$$H \coloneqq \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix},$$

where $x, y \in W$ and $z \in \langle W \cdot W \rangle$, and let $Z \subset H$ be the center of H, that is, the subgroup of H defined by $x = y = 0$. The restriction map

$$H^2(H, M_3(k)) \to H^2(Z, M_3(k))$$

sends $H^2_{\mathrm{neg},F}(H, M_3(k))$ to zero.

Proof. Let $M \coloneqq M_3(k)$. By Lemma 2.1(3), we may assume that F contains all roots of unity of 2-power order. Since H is finite, the conclusion will follow from Theorem 2.2 once we show that $\mathrm{res}^H_Z \mathrm{cor}^S_H(A \cup \chi) = 0$ in $H^2(Z, M)$ for all subgroups $S \subset H$, for all $A \in M^S$ and all $\chi \in H^2(S, \mathbb{Z})$.

Choose a subgroup $S \subset H$, an element $A \in M^S$, and an element $\chi \in H^2(S, \mathbb{Z})$. Letting $\partial\colon H^1(S, \mathbb{Q}/\mathbb{Z}) \to H^2(S, \mathbb{Z})$ be the connecting homomorphism associated to the short exact sequence of S-modules

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Q} \longrightarrow \mathbb{Q}/\mathbb{Z} \longrightarrow 0,$$

we have $\chi = \partial(u)$ for a unique character $u\colon S \to \mathbb{Q}/\mathbb{Z}$. Since Z is an elementary abelian 2-group, there exists a homomorphism $v\colon Z \to \mathbb{Q}/\mathbb{Z}$ which extends the restriction of u to $S \cap Z$. Then the map $\tilde{u}\colon SZ \to \mathbb{Q}/\mathbb{Z}$ defined by $sz \mapsto u(s) + v(z)$ for all $s \in S$ and $z \in Z$ is a well-defined character which extends u. Letting $\tilde{\chi}$ be the image of $\tilde{u}$ in $H^2(SZ, \mathbb{Z})$, we deduce that $\mathrm{res}^{SZ}_S(\tilde{\chi}) = \chi$. By the projection formula, we have

$$\mathrm{cor}^S_H(A \cup \chi) = \mathrm{cor}^{SZ}_H(\mathrm{cor}^S_{SZ}(A \cup \chi)) = \mathrm{cor}^{SZ}_H(N_{SZ/Z}(A) \cup \tilde{\chi}).$$

Therefore, replacing S by SZ, we may assume that $Z \subset S$.

Note that $Z \cap \sigma S \sigma^{-1} = Z$ for every $\sigma \in H$. Hence, by the double coset formula

$$\begin{aligned}
\mathrm{res}^H_Z \mathrm{cor}^S_H(A \cup \chi) &= \sum_\sigma \sigma_*(A \cup \mathrm{res}^S_Z(\chi)) \\
&= \sum_\sigma \sigma_*(A) \cup \sigma_*(\mathrm{res}^S_Z(\chi)) \\
&= \sum_\sigma \sigma_*(A) \cup \mathrm{res}^S_Z(\chi) \\
&= N_{H/S}(A) \cup \mathrm{res}^S_Z(\chi)
\end{aligned}$$

in $H^2(Z, M)$, where σ runs over a set of representatives of H/S. In order to conclude, it remains to show that $N_{H/S}(A) \cup \mathrm{res}^S_Z(\chi) = 0$.

Let $W^\times \coloneqq W \setminus \{0\}$. For every $x \in W^\times$, we define $\sigma_{12}(x) \coloneqq I + xE_{12} \in H$ and $\sigma_{23}(x) \coloneqq I + xE_{23} \in H$. For all $x \in W^\times$, we have

$$M^{\sigma_{12}(x)} = \begin{bmatrix} a & b & c \\ 0 & a & 0 \\ 0 & d & e \end{bmatrix},$$

which is independent of the choice of $x \in W^\times$. Thus, for all $x, y \in W^\times$, we have

$$N_{\sigma_{12}(x)}(M^{\sigma_{12}(y)}) = N_{\sigma_{12}(x)}(M^{\sigma_{12}(x)}) = 0. \tag{5.1}$$

Similarly,

$$N_{\sigma_{23}(x)}(M^{\sigma_{23}(y)}) = N_{\sigma_{23}(x)}(M^{\sigma_{23}(x)}) = 0. \tag{5.2}$$

We split the proof that $N_{H/S}(A) \cup \mathrm{res}^Z_S(\chi) = 0$ in five cases.

(i) Suppose first that $\sigma_{12}(x) \notin S$ for all $x \in W^\times$. Choose two distinct $x, y \in W^\times$, and let K be the subgroup of H generated by S, $\sigma_{12}(x)$, and $\sigma_{12}(y)$. Since $Z \subset S$ and the group H/Z is abelian, we see that S is normal in K and K/S is a Klein group generated by the cosets of $\sigma_{12}(x)$ and $\sigma_{12}(y)$. It follows from (5.1) that

$$N_{K/S}(A) = N_{\sigma_{12}(x)}(N_{\sigma_{12}(y)}(A)) = 0,$$

and hence

$$N_{H/S}(A) = N_{H/K}(N_{K/S}(A)) = 0.$$

(ii) Suppose that $\sigma_{23}(x) \notin S$ for all $x \in W^\times$. The conclusion follows as in case (i), replacing σ_{12} by σ_{23} and (5.1) by (5.2).

(iii) Suppose now that there are $x, y \in W^\times$ such that $\sigma_{12}(x) \notin S$ and $\sigma_{12}(y) \in S$. Let K be the subgroup of H generated by S and $\sigma_{12}(x)$. Then K/S is a cyclic group generated by the coset of $\sigma_{12}(x)$. Since $A \in M^S \subset M^{\sigma_{12}(y)}$, it follows from (5.1) that

$$N_{K/S}(A) = N_{\sigma_{12}(x)}(A) = 0$$

and hence

$$N_{H/S}(A) = N_{H/K}(N_{K/S}(A)) = 0.$$

(iv) Suppose now that there are $x, y \in W^\times$ such that $\sigma_{23}(x) \notin S$ and $\sigma_{23}(y) \in S$. We conclude as in case (iii), replacing σ_{12} by σ_{23} and (5.1) by (5.2).

(v) Finally, suppose that $\sigma_{12}(x)$ and $\sigma_{23}(x)$ belong to S for all $x \in W^\times$. In this case, $S = H$. Then $\mathrm{res}^S_Z(\chi) = 0$ since $Z \subset [S, S]$. □

Proof of Theorem 1.1 for $p = 2$, $|k| > 2$ *and* $n \geq 3$. By Lemma 2.7, it suffices to show that $\mathrm{GLift}(k, n)$ is not negligible over F for all $n \geq 3$. By Lemma 2.8, we may assume that $n = 3$ and, by Lemma 2.1(3), we may suppose that F contains all roots of unity of 2-power order.

Let $W \subset k$ be a finite subgroup such that $|W| > 2$, for example a Klein subgroup. Let $H \subset \mathrm{GL}_3(k)$ and $Z \subset H$ be the corresponding finite subgroups in the statement of Lemma 5.2, and let $\alpha \in H^2(H, M_3(k))$ be the restriction of the class of $\mathrm{GLift}(k, 3)$ to H. By Lemma 2.1(2), it suffices to show that α is not negligible over F. By Lemma 5.1, the restriction of α in $H^2(Z, M_3(k))$ is not zero. By Lemma 5.2, the subgroup $H^2_{\mathrm{neg},F}(H, M_3(k))$ restricts to zero in $H^2(Z, M_3(k))$. Thus $\mathrm{res}^H_Z(\alpha)$ is not negligible over F. By Lemma 2.1(2), we conclude that α is not negligible over F, as desired. □

6. Proof of Theorem 1.1 for $|k| = 2$ and $n \geq 5$

6.1. **Notation.** Throughout this section, we let $G \coloneqq \mathrm{GL}_5(\mathbb{F}_2)$, $U \coloneqq U_5(\mathbb{F}_2)$, and $M \coloneqq M_5(\mathbb{F}_2)$. The group G acts on M by matrix conjugation. For every $A \in M$, we let G_A be the stabilizer of A in G. For every subgroup $H \subset G$, we define

$$\varphi_H \colon M^H \otimes H^2(H, \mathbb{Z}) \xrightarrow{\cup} H^2(H, M) \xrightarrow{\mathrm{cor}} H^2(G, M).$$

For every subgroup $H \subset U_5$ and all $1 \leq i \leq j \leq 5$, we let $u_{ij} \colon H \to \mathbb{Q}/\mathbb{Z}$ be the composition of the (i,j)-th coordinate function $H \to \mathbb{Z}/2\mathbb{Z}$ and the inclusion $\mathbb{Z}/2\mathbb{Z} \hookrightarrow \mathbb{Q}/\mathbb{Z}$. The function u_{ij} is not necessarily a group homomorphism. If it is a homomorphism, then it defines an element $u_{ij} \in H^1(H, \mathbb{Q}/\mathbb{Z})$, and we let $\chi_{ij} \coloneqq \partial(u_{ij}) \in H^2(H, \mathbb{Z})$, where $\partial \colon H^1(H, \mathbb{Q}/\mathbb{Z}) \to H^2(H, \mathbb{Z})$ is the connecting map associated to the sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Q} \longrightarrow \mathbb{Q}/\mathbb{Z} \longrightarrow 0.$$

For every $A \in M$, we let $p_A(x), q_A(x) \in \mathbb{F}_2[x]$ be the characteristic polynomial and the minimal polynomial of A, respectively. Observe that $\deg(p_A(x)) = 5$, that $q_A(x)$ divides $p_A(x)$, and that $p_A(x)$ and $q_A(x)$ have the same irreducible factors.

We let $\Pi\colon S_5 \to G$ be the homomorphism which sends a permutation $\sigma \in S_5$ to the corresponding permutation matrix $\Pi(\sigma)$.

6.2. **Projection formula arguments.** We collect lemmas that will be invoked repeatedly in what follows. Their proofs use the projection formula [NSW08, Proposition 1.5.3(iv)].

Lemma 6.1. *Let $H \subset G$ be a subgroup, let $P \subset H$ be a 2-Sylow subgroup, let $A \in M^H$, let $\chi \in H^2(H, \mathbb{Z})$, and let $\chi' := \mathrm{res}^H_P(\chi) \in H^2(P, \mathbb{Z})$. If $\varphi_P(A \otimes \chi') = 0$, then $\varphi_H(A \otimes \chi) = 0$. In particular, if $\varphi_P = 0$, then $\varphi_H = 0$.*

Proof. Since $2M = 0$ and $[H : P]$ is odd, we have $N_{H/P}(A) = [H : P]A = A$. By the projection formula

$$\mathrm{cor}^H_G(A \cup \chi) = \mathrm{cor}^H_G(N_{H/P}(A) \cup \chi) = \mathrm{cor}^H_G(\mathrm{cor}^P_H(A \cup \chi')) = \mathrm{cor}^P_G(A \cup \chi') = 0. \qquad \square$$

Lemma 6.2. *We have $\varphi_U = 0$.*

Proof. We have $M^U = \langle E_{15} \rangle$ and

$$G_{E_{15}} = \begin{pmatrix} 1 & * & * & * & * \\ 0 & * & * & * & * \\ 0 & * & * & * & * \\ 0 & * & * & * & * \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

We have

$$H^1(U, \mathbb{Q}/\mathbb{Z}) = \mathbb{F}_2 \cdot u_{12} \oplus \mathbb{F}_2 \cdot u_{23} \oplus \mathbb{F}_2 \cdot u_{34} \oplus \mathbb{F}_2 \cdot u_{45},$$

and hence

$$H^2(U, \mathbb{Z}) = \mathbb{F}_2 \cdot \chi_{12} \oplus \mathbb{F}_2 \cdot \chi_{23} \oplus \mathbb{F}_2 \cdot \chi_{34} \oplus \mathbb{F}_2 \cdot \chi_{45}.$$

If $\chi \in \{\chi_{12}, \chi_{23}\}$, define a subgroup $U \subset K \subset G$ as

$$K := \begin{pmatrix} 1 & * & * & * & * \\ 0 & 1 & * & * & * \\ 0 & 0 & 1 & * & * \\ 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & * & * \end{pmatrix}.$$

Then u_{12} and u_{23} extend to K, and hence χ is the restriction of some $\chi' \in H^2(K, \mathbb{Z})$. Observe that $M^K \subset M^U$ and that E_{15} is not K-invariant because K is not contained in $G_{E_{15}}$. We deduce that $M^K = 0$, so that in particular $\mathrm{cor}^U_K(E_{15}) = 0$ and therefore

$$\mathrm{cor}^U_G(E_{15} \cup \chi) = \mathrm{cor}^K_G(\mathrm{cor}^U_K(E_{15} \cup \chi)) = \mathrm{cor}^K_G(N_{U/K}(E_{15}) \cup \chi') = 0.$$

If $\chi \in \{\chi_{34}, \chi_{45}\}$, a similar argument, replacing K by the subgroup

$$K' := \begin{pmatrix} * & * & * & * & * \\ * & * & * & * & * \\ 0 & 0 & 1 & * & * \\ 0 & 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

again shows that $\mathrm{cor}^U_G(E_{15} \cup \chi) = 0$. Thus $\varphi_U = 0$, as desired. $\square$

Lemma 6.3. *Let $H \subset U$ be a subgroup. For all $1 \le i \le 4$ and all $A \in M^H$, we have $\varphi_H(A \otimes \chi_{i,i+1}) = 0$.*

Proof. The $\chi_{i,i+1} \in H^2(H, \mathbb{Z})$ extend to elements of $H^2(U, \mathbb{Z})$. By Lemma 6.2 and the projection formula, we have

$$\mathrm{cor}_G^H(A \cup \chi_{i,i+1}) = \mathrm{cor}_G^U(\mathrm{cor}_U^H(A \cup \chi_{i,i+1})) = \mathrm{cor}_G^U(N_{U/H}(A) \cup \chi_{i,i+1}) = 0. \quad \square$$

Lemma 6.4. *For $i = 2, 3$, let $V_i \coloneqq \mathrm{Ker}[u_{i,i+1} \colon U \to \mathbb{F}_2]$.*

(1) Let $H \subset V_2$ be a subgroup, and let $\chi \in H^2(H, \mathbb{Z})$ be either $\partial(u_{13})$ or $\partial(u_{24})$. For all $A \in M^H$, we have $\varphi_H(A \otimes \chi) = 0$.

(2) Let $H \subset V_3$ be a subgroup, and let $\chi \in H^2(H, \mathbb{Z})$ be either $\partial(u_{24})$ or $\partial(u_{35})$. For all $A \in M^H$, we have $\varphi_H(A \otimes \chi) = 0$.

Proof. We first show that $\varphi_{V_i} = 0$ for $i = 2, 3$. We have $M^{V_i} = \langle E_{15} \rangle = M^U$. By the projection formula, for every $\chi \in H^2(V_i, \mathbb{Z})$ we have

$$\mathrm{cor}_G^{V_i}(E_{15} \cup \chi) = \mathrm{cor}_G^U(\mathrm{cor}_U^{V_i}(E_{15} \cup \chi)) = \mathrm{cor}_G^U(E_{15} \cup \mathrm{cor}_U^{V_i}(\chi)).$$

Now Lemma 6.2 implies that $\varphi_{V_i} = 0$, as claimed.

The coordinate maps $u_{13}, u_{24} \colon V_1 \to \mathbb{F}_2$ and $u_{24}, u_{35} \colon V_2 \to \mathbb{F}_2$ are group homomorphisms. Thus χ as in (1) and (2) is well defined and is the restriction of some $\chi' \in H^2(V_i, \mathbb{Z})$. For every $A \in M^H$, the vanishing of φ_{V_i} implies

$$\mathrm{cor}_G^{V_i}(N_{H/V_i}(A) \cup \chi') = \varphi_{V_i}(N_{H/V_i}(A) \otimes \chi') = 0,$$

and hence by the projection formula

$$\mathrm{cor}_G^H(A \cup \chi) = \mathrm{cor}_G^{V_i}(\mathrm{cor}_{V_i}^H(A \cup \chi)) = \mathrm{cor}_G^{V_i}(N_{H/V_i}(A) \cup \chi') = 0. \quad \square$$

Lemma 6.5. *Let $I \in M$ be the identity matrix. For every subgroup $H \subset G$ and every $\chi \in H^2(H, \mathbb{Z})$, we have $\varphi_H(I \otimes \chi) = 0$.*

Proof. Recall that $\mathrm{GL}_n(\mathbb{F}_2) = \mathrm{SL}_n(\mathbb{F}_2)$ is equal to its derived subgroup for all $n \ge 3$; see for example [MT11, Theorem 24.17]. Thus $G = [G, G]$, and hence $H^2(G, \mathbb{Z}) = 0$. By the projection formula, we conclude that

$$\mathrm{cor}_G^H(I \cup \chi) = I \cup N_{G/H}(\chi) = 0. \quad \square$$

6.3. **The case when A is not conjugate to a Jordan block.**

Proposition 6.6. *Suppose that $A \in M$ is not conjugate to a 5×5 Jordan block. Then $\varphi_{G_A}(A \otimes \chi) = 0$ for all $\chi \in H^2(G_A, \mathbb{Z})$.*

We will prove the conclusion of Proposition 6.6 by a case-by-case analysis:

(1) A is diagonalizable over $\mathbb{F}_2$ (Lemma 6.7),
(2) $p_A(x)$ is square-free (Lemma 6.8),
(3) A is not diagonalizable, but it admits a Jordan form over $\mathbb{F}_2$ (Lemma 6.9),
(4) $p_A(x)$ is not square-free and does not split over $\mathbb{F}_2$ (Lemma 6.10).

By Lemma 2.3, for the proof of Proposition 6.6 it suffices to consider one $A \in M$ for each G-orbit. Moreover, letting $M^0 \subset M$ be the G-submodule of trace-zero matrices, for every $A \in M$ one of A and $I + A$ belongs to M^0, and hence by Lemma 6.5 we may assume that $A \in M^0$.

Lemma 6.7. *If $A \in M$ is diagonalizable over $\mathbb{F}_2$, then $\varphi_{G_A} = 0$.*

Proof. By Lemma 6.5, we may assume that $A \in M^0$. We have

$$G_A \cong \mathrm{GL}_d(\mathbb{F}_2) \times \mathrm{GL}_{5-d}(\mathbb{F}_2)$$

for some $0 \leq d \leq 2$. For all $r \geq 3$ the group $\mathrm{GL}_r(\mathbb{F}_2)$ is equal to its derived subgroup, and hence $H^2(\mathrm{GL}_r(\mathbb{F}_2), \mathbb{Z}) = H^1(\mathrm{GL}_r(\mathbb{F}_2), \mathbb{Q}/\mathbb{Z}) = 0$. Thus, if $d = 0, 1$, we have $H^2(G_A, \mathbb{Z}) = 0$ and hence $\varphi_{G_A} = 0$ in this case. When $d = 2$, up to conjugation

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \qquad G_A = \begin{pmatrix} * & * & * & 0 & 0 \\ * & * & * & 0 & 0 \\ * & * & * & 0 & 0 \\ 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & * & * \end{pmatrix}.$$

Consider the following 2-Sylow subgroup of G_A:

$$P \coloneqq \begin{pmatrix} 1 & * & * & 0 & 0 \\ 0 & 1 & * & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \cong U_3(\mathbb{F}_2) \times U_2(\mathbb{F}_2).$$

By Lemma 6.1, it suffices to show that $\varphi_P = 0$. As

$$H^1(P, \mathbb{Q}/\mathbb{Z}) = \mathbb{F}_2 \cdot u_{12} \oplus \mathbb{F}_2 \cdot u_{23} \oplus \mathbb{F}_2 \cdot u_{45},$$

every character of P extends to U_5, and the conclusion follows from Lemma 6.3. □

Lemma 6.8. *For every $A \in M$ such that $p_A(x)$ is square-free, we have $\varphi_{G_A} = 0$.*

Proof. We view M as a non-commutative $\mathbb{F}_2$-algebra, and for every $A \in M$ we let $Z(A) \subset M$ be the centralizer $\mathbb{F}_2$-subalgebra of A. Then $G_A = Z(A)^\times$.

Suppose that $p_A(x)$ is square-free. Then $p_A(x) = q_A(x)$, and hence A admits a cyclic basis. It follows that $Z(A)$ is equal to the $\mathbb{F}_2$-subalgebra generated by A, and hence $Z(A) \cong \mathbb{F}_2[x]/(p_A(x))$ as $\mathbb{F}_2$-algebras. In particular, $G_A \cong \mathbb{F}_2[x]/(p_A(x))^\times$. Because $p_A(x)$ is square-free, $\mathbb{F}_2[x]/(p_A(x)) \cong F_1 \times \cdots \times F_d$, where $F_i/\mathbb{F}_2$ is a finite field extension for all $1 \leq i \leq d$. Thus $G_A \cong F_1^\times \times \cdots \times F_d^\times$ has odd order. We conclude that $H^2(G_A, \mathbb{Z})[2] = 0$, and hence in particular $\varphi_{G_A} = 0$. □

Lemma 6.9. *Suppose that $A \in M$ is not diagonalizable over $\mathbb{F}_2$, that $p_A(x)$ splits as a product of linear factors in $\mathbb{F}_2[x]$, and that A is not conjugate to a 5×5 Jordan block. Then $\varphi_{G_A}(A \otimes \chi) = 0$ for all $\chi \in H^2(G_A, \mathbb{Z})$.*

Proof. By Lemma 6.5, we may assume that the trace of A is zero. Up to conjugation, we may assume that A is in normal Jordan form. We let $J_r(\lambda)$ be the $r \times r$ Jordan block with eigenvalue $\lambda_i \in \mathbb{F}_2$. More generally, we let $J_{r_1}(\lambda_1) \oplus \cdots \oplus J_{r_d}(\lambda_d)$ be the matrix in Jordan form with i-th Jordan block of size $r_i \geq 1$ and eigenvalue $\lambda_i \in \mathbb{F}_2$.

(i) If $A = J_5(0)$, there is nothing to prove.

(ii) If $A = J_4(0) \oplus J_1(0)$, then

$$G_A = \begin{pmatrix} 1 & a & b & c & d \\ 0 & 1 & a & b & 0 \\ 0 & 0 & 1 & a & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & e & 1 \end{pmatrix}.$$

We replace A by its conjugate by the permutation matrix $\Pi(45)$. Then

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \qquad G_A = \begin{pmatrix} 1 & a & b & d & c \\ 0 & 1 & a & 0 & b \\ 0 & 0 & 1 & 0 & a \\ 0 & 0 & 0 & 1 & e \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

We have $I + E_{15} = [I + E_{14}, I + E_{45}]$. We deduce that $G_A^{\mathrm{ab}} = \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ and $H^1(G_A, \mathbb{Q}/\mathbb{Z}) = (\mathbb{Z}/4\mathbb{Z}) \cdot u \oplus \mathbb{F}_2 \cdot u_{14} \oplus \mathbb{F}_2 \cdot u_{45}$, where $u\colon G_A \to \mathbb{Z}/4\mathbb{Z} \subset \mathbb{Q}/\mathbb{Z}$ is defined as follows. Let $C \subset U_3$ be the subgroup generated by the order 4 element $I + E_{12} + E_{23}$. We have an isomorphism $\rho\colon C \xrightarrow{\sim} \mathbb{Z}/4\mathbb{Z}$ which sends $I + E_{12} + E_{23}$ to $1 + 4\mathbb{Z}$. Then u is the composite of the projection onto the top-left 3×3 square (whose image is equal to C) and ρ. Therefore

$$H^2(G_A, \mathbb{Z}) = (\mathbb{Z}/4\mathbb{Z})\chi \oplus \mathbb{F}_2\chi_{14} \oplus \mathbb{F}_2\chi_{45},$$

where $\chi \coloneqq \partial(u)$.

We first show that $\mathrm{cor}_G^{G_A}(A \cup \chi) = 0$. Let

$$K \coloneqq \begin{pmatrix} 1 & a & b & d & c \\ 0 & 1 & a & 0 & f \\ 0 & 0 & 1 & 0 & a \\ 0 & 0 & 0 & 1 & e \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Then u extends to $u'\colon K \to \mathbb{Z}/4\mathbb{Z}$, with the same definition. We let $\chi' \coloneqq \partial(u')$. Let $\sigma \coloneqq I + E_{25} \in K$. Then K is the internal semidirect product $G_A \rtimes \langle\sigma\rangle$. It follows that $N_{K/G_A}(A) = N_\sigma(A) = E_{15}$. Now the projection formula implies that $\mathrm{cor}_G^{G_A}(A \cup \chi) = \mathrm{cor}_G^K(E_{15} \cup \chi')$. Let

$$L \coloneqq G_{E_{15}} = \begin{pmatrix} 1 & * & * & * & * \\ 0 & * & * & * & * \\ 0 & * & * & * & * \\ 0 & * & * & * & * \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

By the projection formula $\mathrm{cor}_L^K(E_{15} \cup \chi') = E_{15} \cup \mathrm{cor}_L^K(\chi')$. We have

$$H^1(L, \mathbb{Q}/\mathbb{Z}) = \mathbb{F}_2 \cdot u_{12} \oplus \mathbb{F}_2 \cdot u_{45},$$

and so $\varphi_L = 0$ by Lemma 6.3. We conclude that $\mathrm{cor}_G^K(E_{15} \cup \chi') = 0$, and hence in particular $\mathrm{cor}_G^{G_A}(A \cup \chi) = 0$.

The fact that $\mathrm{cor}_G^{G_A}(A \cup \chi_{45}) = 0$ follows from Lemma 6.3. Finally, in order to deal with χ_{14}, we further conjugate A by $\Pi(243)$. Then G_A is sent to

$$\begin{pmatrix} 1 & d & a & b & c \\ 0 & 1 & 0 & 0 & e \\ 0 & 0 & 1 & a & b \\ 0 & 0 & 0 & 1 & a \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

and u_{14} is sent to u_{12}. Now Lemma 6.3 implies that $\mathrm{cor}_G^{G_A}(A \cup \chi_{14}) = 0$.

(iii) If $A = J_3(0) \oplus J_2(0)$, then

$$G_A = \begin{pmatrix} 1 & a & b & c & d \\ 0 & 1 & a & 0 & c \\ 0 & 0 & 1 & 0 & 0 \\ 0 & e & f & 1 & g \\ 0 & 0 & e & 0 & 1 \end{pmatrix}.$$

Conjugate A by $\Pi(2354)$ to get

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \qquad G_A = \begin{pmatrix} 1 & c & a & d & b \\ 0 & 1 & e & h & f \\ 0 & 0 & 1 & c & a \\ 0 & 0 & 0 & 1 & e \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Then $[G_A, G_A]$ contains

$$\begin{aligned} I + E_{14} &= [I + E_{12} + E_{34}, I + E_{13} + E_{35}], \\ I + E_{25} &= [I + E_{23} + E_{45}, I + E_{13} + E_{35}], \\ I + E_{15} &= [I + E_{14}, I + E_{23} + E_{45}], \\ I + E_{13} + E_{24} + E_{35} &= [I + E_{12} + E_{34}, I + E_{23} + E_{45}]. \end{aligned}$$

Thus the abelianization G_A^{ab} may be described as

$$\begin{pmatrix} 1 & c & a & \square & \square \\ 0 & 1 & e & h & \square \\ 0 & 0 & 1 & c & a \\ 0 & 0 & 0 & 1 & e \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{modulo} \quad \begin{pmatrix} 1 & 0 & 1 & \square & \square \\ 0 & 1 & 0 & 1 & \square \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

where the boxes indicate that the corresponding entries are missing. Indeed, G_A^{ab} is a quotient of this group, and on the other hand a simple computation shows that this group is abelian and in fact every element has order 2. Thus $G_A^{\mathrm{ab}} \cong (\mathbb{Z}/2\mathbb{Z})^3$. In fact, we have an isomorphism

$$G_A^{\mathrm{ab}} \xrightarrow{\sim} (\mathbb{Z}/2\mathbb{Z})^3, \qquad \begin{pmatrix} 1 & c & a & \square & \square \\ 0 & 1 & e & h & \square \\ 0 & 0 & 1 & c & a \\ 0 & 0 & 0 & 1 & e \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \mapsto (c, e, a + h + ce).$$

In particular, $H^1(G_A, \mathbb{Q}/\mathbb{Z}) = \mathbb{F}_2 \cdot u_{12} \oplus \mathbb{F}_2 \cdot u_{23} \oplus \mathbb{F}_2 \cdot u$, where

$$u \colon G_A \to \mathbb{Z}/2\mathbb{Z}, \qquad \begin{pmatrix} 1 & c & a & d & b \\ 0 & 1 & e & h & f \\ 0 & 0 & 1 & c & a \\ 0 & 0 & 0 & 1 & e \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \mapsto a + h + ce.$$

We define $\chi := \partial(u) \in H^2(G_A, \mathbb{Z})$. In view of Lemma 6.3, it suffices to show that $\mathrm{cor}_G^{G_A}(A \cup \chi) = 0$. For this, define the subgroup

$$K := \begin{pmatrix} 1 & c & a & d & b \\ 0 & 1 & e & h & f \\ 0 & 0 & 1 & c & g \\ 0 & 0 & 0 & 1 & e \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Then K is the internal semidirect product $G_A \rtimes \langle \sigma \rangle$, where $\sigma := I + E_{35}$. Observe that u extends to a homomorphism $u' \colon K \to \mathbb{Z}/2\mathbb{Z}$, given by the same formula. Let $\chi' := \partial(u')$. We have $N_{K/G_A}(A) = N_\sigma(A) = E_{15}$, and hence by the projection formula $\mathrm{cor}_K^{G_A}(A \cup \chi) = E_{15} \cup \chi'$. This reduces us to showing that $\mathrm{cor}_G^K(E_{15} \cup \chi') = 0$. As in (ii), let

$$L := G_{E_{15}} = \begin{pmatrix} 1 & * & * & * & * \\ 0 & * & * & * & * \\ 0 & * & * & * & * \\ 0 & * & * & * & * \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

By the projection formula $\mathrm{cor}_L^K(E_{15} \cup \chi') = E_{15} \cup \mathrm{cor}_L^K(\chi')$. We have

$$H^1(L, \mathbb{Q}/\mathbb{Z}) = \mathbb{F}_2 \cdot u_{12} \oplus \mathbb{F}_2 \cdot u_{45},$$

and hence $\varphi_L = 0$ by Lemma 6.3. In particular,

$$\mathrm{cor}_G^K(E_{15} \cup \chi') = \mathrm{cor}_G^L(\mathrm{cor}_L^K(E_{15} \cup \chi')) = 0,$$

as desired.

(iv) If $A = J_3(0) \oplus J_1(0) \oplus J_1(0)$, then

$$G_A = \begin{pmatrix} 1 & a & b & c & d \\ 0 & 1 & a & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & e & f & g \\ 0 & 0 & h & i & j \end{pmatrix}.$$

We conjugate A by $\Pi(35)$. Then G_A is sent to

$$\begin{pmatrix} 1 & a & d & c & b \\ 0 & 1 & 0 & 0 & a \\ 0 & 0 & j & i & h \\ 0 & 0 & g & f & e \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

A 2-Sylow subgroup of G_A is

$$P = \begin{pmatrix} 1 & a & d & c & b \\ 0 & 1 & 0 & 0 & a \\ 0 & 0 & 1 & i & h \\ 0 & 0 & 0 & 1 & e \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

We have

$$\begin{aligned}I+E_{15}&=[I+E_{14},I+E_{45}],\\ I+E_{35}&=[I+E_{34},I+E_{45}],\\ I+E_{14}&=[I+E_{13},I+E_{34}].\end{aligned}$$

Then $G_A^{\mathrm{ab}}\cong(\mathbb{Z}/2\mathbb{Z})^4$, so that

$$H^1(G_A,\mathbb{Q}/\mathbb{Z})=\mathbb{F}_2\cdot u_{12}\oplus\mathbb{F}_2\cdot u_{13}\oplus\mathbb{F}_2\cdot u_{34}\oplus\mathbb{F}_2\cdot u_{45}.$$

The conclusion follows from Lemma 6.3 and Lemma 6.4.

(v) If $A=J_2(0)\oplus J_2(0)\oplus J_1(0)$, then

$$G_A=\begin{pmatrix}a&b&c&d&e\\0&a&0&c&0\\f&g&h&i&0\\0&f&0&h&0\\0&j&0&k&l\end{pmatrix}.$$

Conjugate A by $\Pi(2453)$ to get

$$G_A=\begin{pmatrix}a&c&e&b&d\\f&h&0&g&i\\0&0&l&j&k\\0&0&0&a&c\\0&0&0&f&h\end{pmatrix}.$$

A 2-Sylow subgroup of G_A is

$$P:=\begin{pmatrix}1&c&e&b&d\\0&1&0&g&i\\0&0&1&j&k\\0&0&0&1&c\\0&0&0&0&1\end{pmatrix}.$$

The commutator subgroup $[G_A,G_A]$ contains $I+E_{15}$, $I+E_{14}$, $I+E_{25}$, $I+E_{35}$. It follows that $G_A^{\mathrm{ab}}\cong(\mathbb{Z}/2\mathbb{Z})^4$, so that

$$H^1(G_A,\mathbb{Q}/\mathbb{Z})=\mathbb{F}_2\cdot u_{12}\oplus\mathbb{F}_2\cdot u_{13}\oplus\mathbb{F}_2\cdot u_{34}\oplus\mathbb{F}_2\cdot u_{24}.$$

The conclusion follows from Lemma 6.3 and Lemma 6.4.

(vi) If $A=J_2(0)\oplus J_1(0)\oplus J_1(0)\oplus J_1(0)$, then up to conjugation $A=I+E_{15}$, in which case

$$G_A=\begin{pmatrix}1&*&*&*&*\\0&*&*&*&*\\0&*&*&*&*\\0&*&*&*&*\\0&0&0&0&1\end{pmatrix}$$

which contains U. We have $H^1(G_A,\mathbb{Q}/\mathbb{Z})=\mathbb{F}_2\cdot u_{12}\oplus\mathbb{F}_2\cdot u_{45}$. The conclusion follows from Lemma 6.3.

(vii) If $A = J_4(1) \oplus J_1(0)$, then

$$G_A = \begin{pmatrix} 1 & a & b & c & 0 \\ 0 & 1 & a & b & 0 \\ 0 & 0 & 1 & a & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Let

$$K := \begin{pmatrix} 1 & a & b & c & d \\ 0 & 1 & a & b & e \\ 0 & 0 & 1 & a & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Let $\sigma := I + E_{15}$ and $\tau := I + E_{25}$. Then $\langle \sigma, \tau \rangle \cong (\mathbb{Z}/2\mathbb{Z})^2$ is a normal subgroup of U_5 which intersects G_A trivially, and so $K = \langle \sigma, \tau \rangle \rtimes G_A$. In particular, every character of G_A extends to K. By the projection formula, for every $\chi \in H^2(G_A, \mathbb{Z})$ we have

$$\operatorname{cor}_G^{G_A}(A \cup \chi) = \operatorname{cor}_G^K(\operatorname{cor}_K^{G_A}(A \cup \chi)) = \operatorname{cor}_G^{G_A}(N_{K/H}(A) \cup \chi'),$$

where $\chi' \in H^2(K, \mathbb{Z})$ restricts to χ in $H^2(G_A, \mathbb{Z})$. We have $N_\sigma(A) = E_{15}$, so that $N_{K/H}(A) = N_\tau(N_\sigma(A)) = N_\tau(E_{15}) = 0$. Thus $\operatorname{cor}_G^{G_A}(A \cup \chi) = 0$ for all $\chi \in H^2(G_A, \mathbb{Z})$.

(viii) If $A = J_3(1) \oplus J_1(1) \oplus J_1(0)$, then

$$G_A = \begin{pmatrix} 1 & a & b & c & 0 \\ 0 & 1 & a & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & d & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Conjugate A by $\Pi(34)$ to get

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \qquad G_A = \begin{pmatrix} 1 & a & c & b & 0 \\ 0 & 1 & 0 & a & 0 \\ 0 & 0 & 1 & d & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Consider the subgroup

$$K := \begin{pmatrix} 1 & a & c & b & e \\ 0 & 1 & 0 & a & f \\ 0 & 0 & 1 & d & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Let $\sigma := I + E_{15}$ and $\tau := I + E_{25}$. Then $K = \langle \sigma, \tau \rangle \rtimes G_A$, so that in particular every character of G_A extends to K. We have $N_\sigma(A) = E_{15}$ and $N_\tau(E_{15}) = 0$, so that $N_{K/G_A}(A) = N_\tau(N_\sigma(A)) = N_\tau(E_{15}) = 0$. By the projection formula, for all $\chi \in H^2(G_A, \mathbb{Z})$, letting $\chi' \in H^2(K, \mathbb{Z})$ be a class restricting to χ, we have

$$\operatorname{cor}_G^{G_A}(A \cup \chi) = \operatorname{cor}_G^K(\operatorname{cor}_K^{G_A}(A \cup \chi)) = \operatorname{cor}_G^K(N_{K/G_A}(A) \cup \chi') = 0.$$

(ix) If $A = J_2(1) \oplus J_2(1) \oplus J_1(0)$, then

$$G_A = \begin{pmatrix} a & b & c & d & 0 \\ 0 & a & 0 & c & 0 \\ e & f & g & h & 0 \\ 0 & e & 0 & g & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

We conjugate A by $\Pi(23)$. Then

$$A = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \qquad G_A = \begin{pmatrix} a & c & b & d & 0 \\ e & g & f & h & 0 \\ 0 & 0 & a & c & 0 \\ 0 & 0 & e & g & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

A 2-Sylow subgroup of G_A is given by

$$P \coloneqq \begin{pmatrix} 1 & c & b & d & 0 \\ 0 & 1 & f & h & 0 \\ 0 & 0 & 1 & c & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

By Lemma 6.1, it suffices to show that $\mathrm{cor}^P_G(A \cup \chi) = 0$ for all $\chi \in H^2(P, \mathbb{Z})$. Let

$$K \coloneqq \begin{pmatrix} 1 & c & b & d & i \\ 0 & 1 & f & h & j \\ 0 & 0 & 1 & c & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Then $K = \langle \sigma, \tau \rangle \rtimes P$, where $\sigma \coloneqq I + E_{15}$ and $\tau \coloneqq I + E_{25}$. Every character of P extends to K, and hence by the projection formula it suffices to show that $N_{K/P}(A) = 0$. We have $N_\sigma(A) = E_{15}$ and $N_\tau(E_{15}) = 0$, which together imply $N_{K/P}(A) = N_\tau(N_\sigma(A)) = 0$, as desired.

(x) If $A = J_2(1) \oplus J_1(1) \oplus J_1(1) \oplus J_1(0)$, then

$$G_A = \begin{pmatrix} 1 & a & b & c & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & d & e & f & 0 \\ 0 & g & h & i & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

We conjugate A by $\Pi(24)$, so that G_A is replaced by

$$G_A = \begin{pmatrix} 1 & c & b & a & 0 \\ 0 & i & h & g & 0 \\ 0 & f & e & d & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

A 2-Sylow subgroup of G_A is

$$P := \begin{pmatrix} 1 & c & b & a & 0 \\ 0 & 1 & h & g & 0 \\ 0 & 0 & 1 & d & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Thus $P \cong U_4(\mathbb{F}_2)$, and hence $H^1(P, \mathbb{Q}/\mathbb{Z}) = \mathbb{F}_2 \cdot u_{12} \oplus \mathbb{F}_2 \cdot u_{23} \oplus \mathbb{F}_2 \cdot u_{34}$. We conclude by Lemma 6.3.

(xi) If $A = J_3(0) \oplus J_2(1)$, then

$$G_A = \begin{pmatrix} 1 & a & b & 0 & 0 \\ 0 & 1 & a & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & c \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Consider the subgroup

$$K := \begin{pmatrix} 1 & a & b & 0 & 0 \\ 0 & 1 & a & 0 & 0 \\ 0 & 0 & 1 & d & e \\ 0 & 0 & 0 & 1 & c \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Let $\tau := I + E_{35}$ and $\sigma := I + E_{34}$. We have $K = \langle \sigma, \tau \rangle \rtimes G_A$, and hence all characters of G_A extend to K. We have $N_\tau(A) = E_{25} + E_{35}$ and $N_\sigma(E_{25} + E_{35}) = 0$, so that $N_{K/G_A}(A) = N_\sigma(N_\tau(A)) = 0$. By the projection formula, for all $\chi \in H^2(G_A, \mathbb{Z})$, letting $\chi' \in H^2(K, \mathbb{Z})$ be an element restricting to χ, we have

$$\mathrm{cor}_G^{G_A}(A \cup \chi) = \mathrm{cor}_G^K(\mathrm{cor}_K^{G_A}(A \cup \chi)) = \mathrm{cor}_G^K(N_{K/G_A}(A) \cup \chi') = 0.$$

(xii) If $A = J_3(0) \oplus J_1(1) \oplus J_1(1)$, then

$$G_A = \begin{pmatrix} 1 & a & b & 0 & 0 \\ 0 & 1 & a & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & c & d \\ 0 & 0 & 0 & e & f \end{pmatrix}.$$

A 2-Sylow subgroup of G_A is

$$P := \begin{pmatrix} 1 & a & b & 0 & 0 \\ 0 & 1 & a & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & d \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

By Lemma 6.1, it suffices to show that $\mathrm{cor}_G^P(A \cup \chi) = 0$ for all $\chi \in H^2(P, \mathbb{Z})$. We conclude as in (xi).

(xiii) If $A = J_2(0) \oplus J_1(0) \oplus J_2(1)$, then

$$G_A = \begin{pmatrix} 1 & a & b & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & c & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & d \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

We conjugate A by $\Pi(23)$. Then G_A is replaced by

$$G_A = \begin{pmatrix} 1 & b & a & 0 & 0 \\ 0 & 1 & c & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & d \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

We have $H^1(G_A, \mathbb{Q}/\mathbb{Z}) = \mathbb{F}_2 \cdot u_{12} \oplus \mathbb{F}_2 \cdot u_{23} \oplus \mathbb{F}_2 \cdot u_{45}$. We conclude by Lemma 6.3.

(xiv) If $A = J_2(0) \oplus J_1(0) \oplus J_1(1) \oplus J_1(1)$, then

$$G_A = \begin{pmatrix} 1 & b & c & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & d & 1 & 0 & 0 \\ 0 & 0 & 0 & e & f \\ 0 & 0 & 0 & g & h \end{pmatrix}.$$

We conjugate A by $\Pi(23)$. Then G_A becomes

$$G_A = \begin{pmatrix} 1 & c & b & 0 & 0 \\ 0 & 1 & d & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & e & f \\ 0 & 0 & 0 & g & h \end{pmatrix}.$$

A 2-Sylow subgroup of G_A is

$$P := \begin{pmatrix} 1 & c & b & 0 & 0 \\ 0 & 1 & d & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & f \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

We have $H^1(P, \mathbb{Q}/\mathbb{Z}) = \mathbb{F}_2 \cdot u_{12} \oplus \mathbb{F}_2 \cdot u_{23} \oplus \mathbb{F}_2 \cdot u_{45}$. We conclude by Lemma 6.1 and Lemma 6.3.

(xv) If $A = J_1(0) \oplus J_1(0) \oplus J_1(0) \oplus J_2(1)$, then

$$G_A = \begin{pmatrix} a & b & c & 0 & 0 \\ d & e & f & 0 & 0 \\ g & h & i & 0 & 0 \\ 0 & 0 & 0 & 1 & j \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

A 2-Sylow subgroup of G_A is

$$P := \begin{pmatrix} 1 & a & c & 0 & 0 \\ 0 & 1 & b & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & d \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Thus $H^1(P, \mathbb{Q}/\mathbb{Z}) = \mathbb{F}_2 \cdot u_{12} \oplus \mathbb{F}_2 \cdot u_{23} \oplus \mathbb{F}_2 \cdot u_{45}$, and the conclusion follows from Lemma 6.1 and Lemma 6.3. □

Lemma 6.10. *Let $A \in M$ be such that $p_A(x)$ is divisible by a square and does not split as a product of linear factors over $\mathbb{F}_2$. Then $\varphi_{G_A}(A \otimes \chi) = 0$ for all $\chi \in H^2(G_A, \mathbb{Z})$.*

Proof. By Lemma 6.5, we may assume that the trace of A is equal to 0. Write $p_A(x) = p_1(x)p_2(x)^2$, where $p_1(x)$ is square-free. Then $\deg(p_2(x)) \in \{1, 2\}$, and hence

$$p_2(x) \in \{x^2, (x+1)^2, x^2(x+1)^2, (x^2+x+1)^2, x^4\}.$$

We exclude $x^2(x+1)^2$ and x^4 because by assumption $p_A(x)$ does not split over $\mathbb{F}_2$. Thus

$$p_2(x) \in \{x^2, (x+1)^2, (x^2+x+1)^2\}.$$

Since the trace of A is zero, the sum of the roots of $p_A(x)$ in $\overline{\mathbb{F}}_2$ is equal to zero. As each root of $p_2(x)^2$ in $\overline{\mathbb{F}}_2$ has even multiplicity, we deduce that the sum of the roots of $p_1(x)$ in $\overline{\mathbb{F}}_2$ must be equal to 0, so that

$$p_1(x) = x^d + a_{d-2}x^{d-2} + \cdots + a_1 x + a_0.$$

Therefore, if $p_2(x) = (x^2+x+1)^2$, then $p_1(x) = x$. If $p_2(x) \in \{x^2, (x+1)^2\}$, then $p_1(x) = x^3 + a_1 x + a_0$ for some $a_i \in \mathbb{F}_2$, but x^3 and $x^3 + x = (x+1)x^2$ must be excluded because by assumption $p_A(x)$ does not split over $\mathbb{F}_2$, and hence $p_1(x)$ belongs to $\{x^3+x+1, (x+1)(x^2+x+1)\}$ in this case. All in all, the possibilities for $p_A(x)$ are

$$(x^3+x+1)x^2, \quad (x^3+x+1)(x+1)^2, \quad (x^2+x+1)^2 x,$$

$$(x^2+x+1)x^2(x+1), \quad (x^2+x+1)(x+1)^3.$$

We now prove Lemma 6.10 by a case-by-case analysis.

(i) If $p_A(x) = (x^3+x+1)x^2$ and $q_A(x) = (x^3+x+1)x$, then up to conjugation

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \qquad G_A = \begin{pmatrix} a+c & a & b & 0 & 0 \\ b & c & a+b & 0 & 0 \\ a & b & c & 0 & 0 \\ 0 & 0 & 0 & d & e \\ 0 & 0 & 0 & f & g \end{pmatrix}.$$

The top-left 3×3 corner is isomorphic to $(\mathbb{F}_2[x]/(x^3+x+1))^\times \cong \mathbb{F}_8^\times \cong \mathbb{Z}/7\mathbb{Z}$. Therefore $G_A \cong \mathbb{Z}/7\mathbb{Z} \times \mathrm{GL}_2(\mathbb{F}_2)$. In particular, $I + E_{45} \in G_A$ generates a 2-Sylow subgroup of G_A. The conclusion follows from Lemma 6.1 and Lemma 6.3.

(ii) If $p_A(x) = q_A(x) = (x^3+x+1)x^2$, then up to conjugation

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \qquad G_A = \begin{pmatrix} a+c & a & b & 0 & 0 \\ b & c & a+b & 0 & 0 \\ a & b & c & 0 & 0 \\ 0 & 0 & 0 & 1 & d \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

We conclude as in (i).

(iii) If $p_A(x) = (x^3+x+1)(x+1)^2$ and $q_A(x) = (x^3+x+1)(x+1)$, then G_A is as in (i), and we conclude as in (i).

(iv) If $p_A(x) = q_A(x) = (x^3+x+1)(x+1)^2$, then G_A is as in (ii), and we conclude as in (ii).

(v) If $p_A(x) = (x^2+x+1)^2x$ and $q_A(x) = (x^2+x+1)x$, then up to conjugation

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \qquad G_A = \begin{pmatrix} a+b & a & c+d & c & 0 \\ a & b & c & d & 0 \\ e+f & e & g+h & g & 0 \\ e & f & g & h & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

The subring

$$\begin{bmatrix} a+b & a \\ a & b \end{bmatrix} \subset M_2(\mathbb{F}_2)$$

is isomorphic to $\mathbb{F}_4$. In particular, it is commutative, and its unit group is cyclic of order 3. Let $z \in \mathbb{F}_4$ be such that z^2+z+1, so that $\mathbb{F}_4 = \mathbb{F}_2 \cdot 1 \oplus \mathbb{F}_2 \cdot z$ as an $\mathbb{F}_2$-vector space. This identification yields an inclusion $\mathrm{GL}_2(\mathbb{F}_4) \hookrightarrow \mathrm{GL}_4(\mathbb{F}_2)$ with image G_A, where we also identify G_A with its image under the injective homomorphism $G_A \hookrightarrow \mathrm{GL}_4(\mathbb{F}_2)$ given by the top-left 4×4 square. Since the natural $\mathrm{GL}_2(\mathbb{F}_4)$-action on $\mathbb{F}_4^2 \setminus \{0\}$ is transitive, G_A acts transitively on $\mathbb{F}_2^4 \setminus \{0\}$. The G_A-stabilizer of $e_2 \in \mathbb{F}_2^4$ is

$$S := \begin{pmatrix} 1 & 0 & c+d & c & 0 \\ 0 & 1 & c & d & 0 \\ 0 & 0 & g+h & g & 0 \\ 0 & 0 & g & h & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

As $[G_A : S] = |\mathbb{F}_2^4 \setminus \{0\}| = 2^4 - 1$ is odd, a 2-Sylow subgroup of S is also a 2-Sylow subgroup of G_A. Therefore a 2-Sylow subgroup of G_A is

$$P := \begin{pmatrix} 1 & 0 & c+d & c & 0 \\ 0 & 1 & c & d & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

By Lemma 6.1, it suffices to show that $\varphi_P = 0$. We have $P \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, and hence $H^1(P, \mathbb{Q}/\mathbb{Z}) = \mathbb{F}_2 \cdot u_{13} \oplus \mathbb{F}_2 \cdot u_{23}$. The conclusion follows from Lemma 6.3 and Lemma 6.4.

(vi) If $p_A(x) = q_A(x) = (x^2+x+1)^2x$, then up to conjugation

$$A = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \qquad G_A = \begin{pmatrix} a+b & a & c+d & c & 0 \\ a & b & c & d & 0 \\ 0 & 0 & a+b & a & 0 \\ 0 & 0 & a & b & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

The unique 2-Sylow subgroup of G_A is

$$P \coloneqq \begin{pmatrix} 1 & 0 & c+d & c & 0 \\ 0 & 1 & c & d & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

We conclude as in (v). Indeed, by Lemma 6.1, it suffices to show that $\varphi_P = 0$. We have $H^1(P, \mathbb{Q}/\mathbb{Z}) = \mathbb{F}_2 \cdot u_{13} \oplus \mathbb{F}_2 \cdot u_{23}$, and the conclusion follows from Lemma 6.3 and Lemma 6.4.

(vii) If $p_A(x) = (x^2+x+1)x^2(x+1)$ and $q_A(x) = (x^2+x+1)x(x+1)$, then up to conjugation

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \qquad G_A = \begin{pmatrix} a+b & a & 0 & 0 & 0 \\ a & b & 0 & 0 & 0 \\ 0 & 0 & c & d & 0 \\ 0 & 0 & e & g & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Then $G_A \cong \mathbb{Z}/3\mathbb{Z} \times \mathrm{GL}_2(\mathbb{F}_2)$, and the unique 2-Sylow subgroup of G_A is

$$P \coloneqq \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & d & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

We conclude by Lemma 6.1 and Lemma 6.3.

(viii) If $p_A(x) = q_A(x) = (x^2+x+1)x^2(x+1)$, then up to conjugation

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \qquad G_A = \begin{pmatrix} a+b & a & 0 & 0 & 0 \\ a & b & 0 & 0 & 0 \\ 0 & 0 & 1 & c & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Thus $G_A \cong \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ and the unique 2-Sylow subgroup of G_A is generated by $I + E_{34}$. We conclude by Lemma 6.1 and Lemma 6.3.

(ix) If $p_A(x) = (x^2+x+1)(x+1)^3$ and $q_A(x) = (x^2+x+1)(x+1)$, then up to conjugation

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \qquad G_A = \begin{pmatrix} a+b & a & 0 & 0 & 0 \\ a & b & 0 & 0 & 0 \\ 0 & 0 & c & d & e \\ 0 & 0 & f & g & h \\ 0 & 0 & i & j & k \end{pmatrix}.$$

Thus $G_A \cong \mathbb{Z}/3\mathbb{Z} \times \mathrm{GL}_3(\mathbb{F}_2)$, and hence $H^2(G_A, \mathbb{Z})[2] = 0$.

(x) If $p_A(x) = (x^2+x+1)(x+1)^3$ and $q_A(x) = (x^2+x+1)(x+1)^2$, then up to conjugation

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \qquad G_A = \begin{pmatrix} a+b & a & 0 & 0 & 0 \\ a & b & 0 & 0 & 0 \\ 0 & 0 & 1 & c & d \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & e & 1 \end{pmatrix}.$$

We conjugate A by $\Pi(45)$. Then G_A is replaced by

$$G_A = \begin{pmatrix} a+b & a & 0 & 0 & 0 \\ a & b & 0 & 0 & 0 \\ 0 & 0 & 1 & d & c \\ 0 & 0 & 0 & 1 & e \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Then $G_A \cong \mathbb{Z}/3\mathbb{Z} \times U_3(\mathbb{F}_2)$, and the unique 2-Sylow subgroup of G_A is

$$P \coloneqq \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & d & c \\ 0 & 0 & 0 & 1 & e \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Thus $P \cong U_3(\mathbb{F}_2)$ and in particular $H^1(P, \mathbb{Q}/\mathbb{Z}) = \mathbb{F}_2 \cdot u_{34} \oplus \mathbb{F}_2 \cdot u_{45}$. We conclude by Lemma 6.1 and Lemma 6.3.

(xi) If $p_A(x) = q_A(x) = (x^2+x+1)(x+1)^3$, then up to conjugation

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \qquad G_A = \begin{pmatrix} a+b & a & 0 & 0 & 0 \\ a & b & 0 & 0 & 0 \\ 0 & 0 & 1 & c & d \\ 0 & 0 & 0 & 1 & c \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Then $G_A \cong \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$, and the unique 2-Sylow subgroup of G_A is

$$P \coloneqq \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & c & d \\ 0 & 0 & 0 & 1 & c \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

By Lemma 6.1, it suffices to show that $\mathrm{cor}_G^P(A \cup \chi) = 0$ for all $\chi \in H^2(P, \mathbb{Z})$. Let

$$K \coloneqq \begin{pmatrix} 1 & e & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & c & d \\ 0 & 0 & 0 & 1 & c \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

and let $\sigma \coloneqq I + E_{12} \in K$, so that $K = \langle \sigma, P \rangle \cong \mathbb{Z}/2\mathbb{Z} \times P$. Every character of P extends to K. Let $A' \coloneqq N_\sigma(A) = E_{11} + E_{22} \in M^K$. By the projection formula, for every $\chi \in H^2(P, \mathbb{Z})$, letting $\chi' \in H^2(K, \mathbb{Z})$ be a class restricting to χ, we have

$$\mathrm{cor}_G^P(A \cup \chi) = \mathrm{cor}_G^K(A' \cup \chi').$$

Because $A' \in M^K$, we have $K \subset G_{A'}$, and so by the projection formula

$$\mathrm{cor}_G^K(A' \cup \chi') = \mathrm{cor}_G^{G_{A'}}(A' \cup N_{G_{A'}/K}(\chi')).$$

Since A' is diagonal, the conclusion follows from Lemma 6.7. □

6.4. **Restriction to a Klein subgroup.** Let

$$S := \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad T := \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$$

be matrices in $M_2(\mathbb{F}_2)$. Let $n \geq 4$. We will write $n \times n$ matrices as 3×3 matrices according to the partition $n = 2 + 2 + (n-4)$. The matrices

$$\sigma := \begin{pmatrix} I & S & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{pmatrix}, \quad \tau := \begin{pmatrix} I & T & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{pmatrix}$$

commute and generate a Klein subgroup $Z \subset \mathrm{GL}_n(\mathbb{F}_2)$.

Proposition 6.11. *For every $n \geq 4$, the class of $\mathrm{GLift}(\mathbb{F}_2, n)$ restricts to a non-trivial class in $H^2(Z, M_n(\mathbb{F}_2))$, where $Z = \langle \sigma, \tau \rangle \subset \mathrm{GL}_n(\mathbb{F}_2)$ is the Klein subgroup defined above.*

Proof. Let $M := M_n(\mathbb{F}_2)$. Let

$$\tilde{S} := \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad \tilde{T} := \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$$

be matrices in $M_2(\mathbb{Z}/4\mathbb{Z})$, and define

$$\tilde{\sigma} := \begin{pmatrix} I & \tilde{S} & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{pmatrix}, \qquad \tilde{\tau} := \begin{pmatrix} I & \tilde{T} & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{pmatrix}$$

in $\mathrm{GL}_n(\mathbb{Z}/4\mathbb{Z})$. Then $\tilde{\sigma}$ and $\tilde{\tau}$ commute, $\tilde{\sigma}^{-2} = I + 2s$ and $\tilde{\tau}^2 = I + 2t$, where

$$s := \begin{pmatrix} 0 & S & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad t := \begin{pmatrix} 0 & T & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Suppose by contradiction that $\mathrm{GLift}(\mathbb{F}_2, n)$ restricts to the trivial class in $H^2(Z, M)$. Then, by Lemma 2.9, there are $U, V \in M$ such that:

$$N_\sigma(U) = s, \quad N_\tau(V) = t, \quad N_\tau(U) = N_\sigma(V).$$

We have:

$$N_\sigma(U) = U + \sigma U \sigma = \begin{bmatrix} * & U_{11} + U_{21} + U_{22} & * \\ * & U_{21} & * \\ * & * & * \end{bmatrix},$$

and

$$N_\tau(V) = V + \tau U \tau = \begin{bmatrix} * & V_{11}T + TV_{21}T + TV_{22} & * \\ * & TV_{21} & * \\ * & * & * \end{bmatrix},$$

hence the equation $N_\sigma(U) = s$ implies that $U_{21} = 0$ and

$$U_{11} + U_{22} = I. \tag{6.1}$$

Similarly, the equation $N_\tau(V) = t$ implies that $V_{21} = 0$ and

$$V_{11}T + TV_{22} = T. \tag{6.2}$$

The equation $N_\tau(U) = N_\sigma(V)$ implies

$$U_{11}T + TU_{22} = V_{11} + V_{22}. \tag{6.3}$$

Plugging in $U_{22} = I + U_{11}$ from (6.1) into (6.3) and then V_{22} from (6.3) into (6.2), we get:

$$T(V_{11} + TU_{11}) + (V_{11} + TU_{11})T = T + T^2 = I.$$

Note that the equation $TX + XT = I$ has no solutions in $M_2(\mathbb{F}_2)$, contradicting the existence of U and V, as desired. □

6.5. **The case when A is conjugate to a Jordan block.**

Proposition 6.12. *Suppose that $A \in M$ is conjugate to a 5×5 Jordan block. Then the class of* $\mathrm{GLift}(\mathbb{F}_2, 5)$ *is not in the image of φ_{G_A}.*

Proof. By Lemma 6.5, we may assume that the trace of A is 0, and hence, up to conjugation, that A is the nilpotent 5×5 Jordan block $N = J_5(0)$. We have

$$G_A = \begin{pmatrix} 1 & a & b & c & d \\ 0 & 1 & a & b & c \\ 0 & 0 & 1 & a & b \\ 0 & 0 & 0 & 1 & a \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Thus $G_A \cong (\mathbb{F}_2[x]/(x^5))^\times \cong \mathbb{Z}/8\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, where the factor $\mathbb{Z}/8\mathbb{Z}$ is generated by $I + N$ and the factor $\mathbb{Z}/2\mathbb{Z}$ is generated by $I + N^3 = I + E_{14} + E_{25}$.

Define $u, v \in H^1(G_A, \mathbb{Q}/\mathbb{Z})$ by

$$u(I + N) = 0, \quad u(I + N^3) = 1/2, \quad v(I + N) = 1/8, \quad v(I + N^3) = 1/2,$$

and let $\chi := \partial(u)$ and $\psi := \partial(v)$ in $H^2(G_A, \mathbb{Z})$. We have

$$H^2(G_A, \mathbb{Z}) = (\mathbb{Z}/2\mathbb{Z}) \cdot \chi \oplus (\mathbb{Z}/8\mathbb{Z}) \cdot \psi.$$

We first show that $\mathrm{cor}_G^{G_A}(A \cup \chi) = 0$. Consider the subgroup

$$K := \begin{pmatrix} 1 & a & b & d & f \\ 0 & 1 & a & c & e \\ 0 & 0 & 1 & a & c \\ 0 & 0 & 0 & 1 & a \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Then $G_A \subset K$. We claim that u extends to an element of $H^1(K, \mathbb{Q}/\mathbb{Z})$. For this, let $\overline{K}$ be the quotient of K by the subgroup generated by $I + E_{13}, I + E_{14}, I + E_{15}$:

$$\overline{K} = \begin{pmatrix} 1 & a & \square & \square & \square \\ 0 & 1 & a & c & e \\ 0 & 0 & 1 & a & c \\ 0 & 0 & 0 & 1 & a \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Then $\overline{K} \cong \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, where the $\mathbb{Z}/4\mathbb{Z}$ is generated by the coset of $I + N$, and the $\mathbb{Z}/2\mathbb{Z}$ is generated by the coset of $I + N^3$. It follows that we may define $\overline{u} \in H^1(\overline{K}, \mathbb{Q}/\mathbb{Z})$ by sending the coset of $I + N$ to 0 and the coset of $I + N^3$ to $1/2$. Letting $u' \in H^1(K, \mathbb{Q}/\mathbb{Z})$ be the composition of the quotient map $K \to \overline{K}$ and $\overline{u}$, we see that u' restricts to u on G_A, as claimed. It follows that $\chi' := \partial(u') \in H^2(K, \mathbb{Z})$ extends χ.

Let $\sigma := I + E_{14}$ and $\tau := I + E_{13}$. Then $N_\sigma(A) = E_{15}$ and $N_\tau(E_{15}) = 0$, so that $N_{K/G_A}(A) = N_\tau(N_\sigma(A)) = 0$. By the projection formula,

$$\mathrm{cor}_G^{G_A}(A \cup \chi) = \mathrm{cor}_G^K(\mathrm{cor}_K^{G_A}(A \cup \chi)) = \mathrm{cor}_G^K(N_{K/G_A}(A) \cup \chi') = 0.$$

It remains to show that $\mathrm{cor}_G^{G_A}(A \cup \psi) = 0$. For this, let $Z := \langle \sigma, \tau \rangle \subset U$ be the Klein subgroup of Proposition 6.11. By Proposition 6.11, it suffices to show that $(\mathrm{res}_Z^G \circ \mathrm{cor}_G^{G_A})(A \cup \psi) = 0$. The double coset formula reads

$$\mathrm{res}_Z^G \circ \mathrm{cor}_G^{G_A} = \sum_{g \in R} \mathrm{cor}_Z^{Z \cap gG_Ag^{-1}} \circ g_* \circ \mathrm{res}^{G_A}_{G_A \cap g^{-1}Zg}, \tag{6.4}$$

where $R \subset G$ is a set of representatives for $Z \backslash G / G_A$. The Jordan normal form of $I + N^4$ is $I + E_{12}$, while the Jordan normal form of $\sigma, \tau, \sigma\tau$ is $I + E_{12} + E_{34}$. Thus $g(I + N^4)g^{-1}$ does not belong to Z, for any g. It follows that there are three mutually exclusive possibilities for $Z \cap gG_Ag^{-1}$: either it is trivial, or it is generated by $I + N^3$, or it is generated by $\rho := I + N^3 + N^4$. In the first two cases, the restriction of v to $Z \cap gG_Ag^{-1}$ is zero, and hence the term in (6.4) corresponding to g is zero. Thus (6.4) reduces to

$$\mathrm{res}_Z^G \circ \mathrm{cor}_G^{G_A} = \sum_{g \in S} \mathrm{cor}_Z^{Z \cap gG_Ag^{-1}} \circ g_* \circ \mathrm{res}^{G_A}_{G_A \cap g^{-1}Zg}, \tag{6.5}$$

where $S \subset R$ is the subset of those g such that $g\rho g^{-1} \in Z$. We have $S = S_\sigma \coprod S_\tau \coprod S_{\sigma\tau}$, where by definition g belongs to S_σ (resp. S_τ, $S_{\sigma\tau}$) if and only if $g\rho g^{-1}$ is equal to σ (resp. τ, $\sigma\tau$).

For all $g \in S_\sigma$, the subgroup $Z \cap gG_Ag^{-1}$ is equal to $\langle\sigma\rangle$. Moreover, $g_*(v)$ is the non-trivial element in $H^1(\langle\sigma\rangle, \mathbb{Q}/\mathbb{Z})$, and hence $g_*(\psi) = \partial(g_*(v))$ is the unique non-trivial element in $H^2(\langle\sigma\rangle, \mathbb{Z})$. Let $\psi_\sigma \in H^2(Z, \mathbb{Z})$ which extends $g_*(\psi)$ for $g \in S_\sigma$. By the projection formula, for all $g \in S_\sigma$ we have

$$\begin{aligned}(\mathrm{cor}_Z^{Z \cap gG_Ag^{-1}} \circ g_* \circ \mathrm{res}^{G_A}_{G_A \cap g^{-1}Zg})(A \cup \psi) &= (\mathrm{cor}_Z^{\langle\sigma\rangle} \circ g_* \circ \mathrm{res}^{G_A}_{\langle\rho\rangle})(A \cup \psi) \\ &= \mathrm{cor}_Z^{\langle\sigma\rangle}(g_*(A) \cup g_*(\psi)) \\ &= N_{Z/\langle\sigma\rangle}(g_*(A)) \cup \psi_\sigma \\ &= (gag^{-1} + \tau gag^{-1}\tau^{-1}) \cup \psi_\sigma.\end{aligned}$$

Therefore

$$\sum_{g \in S_\sigma} (\mathrm{cor}_Z^{Z \cap gG_Ag^{-1}} \circ g_* \circ \mathrm{res}^{G_A}_{G_A \cap g^{-1}Zg})(A \cup \psi) = \left(\sum_{g \in S_\sigma} (gag^{-1} + \tau gag^{-1}\tau^{-1})\right) \cup \psi_\sigma.$$

Similarly,

$$\sum_{g \in S_\tau} (\mathrm{cor}_Z^{Z \cap gG_Ag^{-1}} \circ g_* \circ \mathrm{res}^{G_A}_{G_A \cap g^{-1}Zg})(A \cup \psi) = \left(\sum_{g \in S_\tau} (gag^{-1} + \sigma gag^{-1}\sigma^{-1})\right) \cup \psi_\tau,$$

$$\sum_{g \in S_{\sigma\tau}} (\mathrm{cor}_Z^{Z \cap gG_Ag^{-1}} \circ g_* \circ \mathrm{res}^{G_A}_{G_A \cap g^{-1}Zg})(A \cup \psi) = \left(\sum_{g \in S_{\sigma\tau}} (gag^{-1} + \sigma gag^{-1}\sigma^{-1})\right) \cup \psi_{\sigma\tau},$$

where φ_τ (resp. $\varphi_{\sigma\tau}$) is an element of $H^2(Z, \mathbb{Z})$ extending $g_*(\psi)$ for all $g \in S_\tau$ (resp. $g \in S_{\sigma\tau}$). In view of (6.5), the proof will be complete once we show that the three sums

$$\sum_{g\in S_\sigma}(gag^{-1}+\tau gag^{-1}\tau^{-1}),\quad \sum_{g\in S_\tau}(gag^{-1}+\sigma gag^{-1}\sigma^{-1}),\quad \sum_{g\in S_{\sigma\tau}}(gag^{-1}+\sigma gag^{-1}\sigma^{-1})$$

are zero.

For all $g\in S_\sigma$, we have $g\rho=\sigma g$, hence $\sigma gG_A=g\rho G_A=gG_A$, and so

$$ZgG_A=gG_A\cup\sigma gG_A\cup\tau gG_A\cup\sigma\tau gG_A=gG_A\cup\tau gG_A=\langle\tau\rangle\, gG_A.$$

In other words,

$$Z\backslash(ZS_\sigma G_A)/G_A=\langle\tau\rangle\backslash(S_\sigma G_A)/G_A.$$

Note that τ acts without fixed points on $(S_\sigma G_A)/G_A$. Indeed, suppose that $\tau gG_A=gG_A$ for some $g\in S_g$. Then $g^{-1}\tau g\in G_A$. We also have $g^{-1}\sigma g=\rho\in G_A$, and hence $g^{-1}Zg\subset G_A$. As Z and the 2-torsion subgroup $G_A[2]\subset G_A$ have the same order, equal to 4, this implies that $g^{-1}Zg=G_A[2]$, contradicting the fact that $I+N^4$ is not conjugate to any element of Z. We obtain

$$\sum_{g\in S_\sigma}(gag^{-1}+\tau gag^{-1}\tau^{-1})=\sum_g gag^{-1},$$

where the second sum is taken over a set of representatives g of the cosets in $(S_\sigma G_A)/G_A$. Let $C\subset G$ be the centralizer of ρ. Observe that $G_A\subset C$: indeed, a matrix commuting with $A=I+N$ commutes with any polynomial in N such as ρ. Moreover, $S_\sigma G_A=g_0C$ for some $g_\sigma\in S_\sigma$. It follows that the above sum is conjugate via g_0 to $N_{C/G_A}(A)$. The same argument shows that the second and third sums are conjugate to $N_{C/G_A}(A)$ via appropriate $g_\tau\in S_\tau$ and $g_{\sigma\tau}\in S_{\sigma\tau}$, respectively. It remains to show that $N_{C/G_A}(A)=0$. Consider again the subgroup

$$K\coloneqq\begin{pmatrix}1&a&b&d&f\\0&1&a&c&e\\0&0&1&a&c\\0&0&0&1&a\\0&0&0&0&1\end{pmatrix}.$$

Then $G_A\subset K\subset C$, and hence it suffices to show that $N_{K/G_A}(A)=0$. Let $\mu\coloneqq I+E_{14}$ and $\nu\coloneqq I+E_{13}$. Then G_A is normal in K and K/G_A is a Klein group generated by the cosets of μ and ν. We have $N_\mu(A)=I+E_{15}$ and $N_\nu(I+E_{15})=0$, so that $N_{K/G_A}=N_\nu(N_\mu(A))=0$, as desired. □

6.6. **End of Proof of Theorem 1.1.**

Proof of Theorem 1.1 when $|k|=2$ *and* $n\geq 5$. By Lemma 2.7, it suffices to show that $\mathrm{GLift}(\mathbb{F}_2,n)$ is not negligible over F, and by Lemma 2.8, we may assume that $n=5$. By Lemma 2.1(3), we may also assume that F contains all primitive roots of unity of 2-power order.

By Proposition 6.6 and Proposition 6.12, the class of $\mathrm{GLift}(\mathbb{F}_2,5)$ does not belong to the subgroup of $H^2(\mathrm{GL}_5(\mathbb{F}_2),M_5(2))$ generated by the images of the maps φ_H, where H ranges over all subgroups of $\mathrm{GL}_5(\mathbb{F}_2)$. Now Theorem 2.2 implies that the class of $\mathrm{GLift}(\mathbb{F}_2,5)$ is not negligible over F. □

6.7. **Explicit non-liftable Galois representations.** Let F be a field. Let H be a finite group, let V be a faithful finite-dimensional F-linear representation of H over F. We view V as an affine space over F, we let $F(V)$ be the function field of V, and we let $F(V)^H$ be the H-fixed subfield. The field extension $F(V)/F(V)^H$ is Galois with Galois group H. A choice of separable closure of $F(V)^H$ containing $F(V)$ determines a surjective homomorphism $\rho\colon \Gamma_{F(V)^H} \to H$. We say that a pair (K, ρ), where K/F is a field extension and $\rho\colon \Gamma_K \to H$ is a homomorphism, is *generic for H over F* if there exists a faithful finite-dimensional F-linear representation V of H such that $K = F(V)^H$ and ρ is induced by the H-Galois extension $F(V)/F(V)^H$. Of course, a generic pair for H over F always exists.

Proposition 6.13. *Let H be a finite group, let A be a H-module, let (K, ρ) be a generic pair for H over F, and consider a group extension (1.1). The class of (1.1) is negligible over F if and only if ρ lifts to G.*

Proof. See [GM22, Proposition 2.1]. □

For all positive integers n and fields k of characteristic $p > 0$ such that the class of $\mathrm{GLift}(k, n)$ is not negligible over F, using Proposition 6.13 we now exhibit field extensions K/F and continuous homomorphisms $\rho\colon \Gamma_K \to \mathrm{GL}_n(k)$ which do not lift to $\Gamma_K \to \mathrm{GL}_n(W_2(k))$. Indeed, one may take a generic pair (K, ρ) for H over F, where H is the finite subgroup of $\mathrm{GL}_n(k)$ given below. We may assume that $\mathrm{char}(F) \neq p$, since $\mathrm{GLift}(k, n)$ is otherwise negligible over F.

- If $p > 2$ and $n \geq 3$, we may take $H = \mathrm{GL}_3(\mathbb{F}_p)$, embedded in the top-left 3×3 block of $\mathrm{GL}_n(k)$; see the proofs of Lemma 4.1 and from Lemma 2.8.
- If $p = 2$, $|k| > 2$, $n \geq 3$, we may take $H \subset \mathrm{GL}_3(k) \subset \mathrm{GL}_n(k)$ to be the subgroup of $\mathrm{GL}_3(k)$ appearing in the statement of Lemma 5.2, where $\mathrm{GL}_3(k) \subset \mathrm{GL}_n(k)$ is the top-left 3×3 block; see the proofs of Lemma 5.2 and Lemma 2.8.
- If $p = 2$, $|k| = 2$ and $n \geq 5$, we may take $H = \mathrm{GL}_5(\mathbb{F}_2)$, embedded in $\mathrm{GL}_n(\mathbb{F}_2)$ as the top-left 5×5 corner; see the proof of Lemma 2.8.

7. Splitting of $\mathrm{Lift}(k, n)$

For completeness, we determine all cases when the sequence $\mathrm{Lift}(k, n)$ is split.

Theorem 7.1. *Let k be a field of characteristic $p > 0$ and let $n > 0$ be an integer. The sequence $\mathrm{GLift}(k, n)$ is split if and only if one of the following holds:*

- $n = 1$*;*
- $n = 2$ *and* $|k| \leq 3$*;*
- $n = 3$ *and* $|k| = 2$.

Proof. We first show that $\mathrm{GLift}(k, n)$ splits in the cases listed above.

(i) If $n = 1$, a splitting of the map $\pi\colon W_2(k)^\times \to k^\times$ is given by the Teichmüller lift, that is, the group homomorphism $\tau\colon k^\times \to W_2(k)^\times$ given by $\tau(x) = (x, 0)$.

In all remaining cases, k is finite, and hence the sequence $\mathrm{GLift}(k, n)$ is split if and only if its restriction to the p-Sylow subgroup $U_n(k) \subset \mathrm{GL}_n(k)$ is split. We will construct splittings over $U_n(k)$.

(ii) If $n = 2$ and $k = \mathbb{F}_2$, a splitting is given by

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \mapsto \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}.$$

(iii) If $n = 2$ and $k = \mathbb{F}_3$, a splitting is given by

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \mapsto \begin{pmatrix} 1 & 1 \\ -3 & -2 \end{pmatrix}.$$

(iv) If $n = 3$ and $k = \mathbb{F}_2$, a splitting is given by

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \mapsto \begin{pmatrix} -1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \mapsto \begin{pmatrix} -1 & 0 & 0 \\ 2 & 1 & -1 \\ 0 & 0 & -1 \end{pmatrix}.$$

Indeed, letting $\sigma_1, \sigma_2 \in \mathrm{GL}_3(\mathbb{Z}/4\mathbb{Z})$ be the images of $I + E_{12}, I + E_{23}$, and letting $\tau \coloneqq [\sigma_1, \sigma_2]$, it suffices to check that

$$\sigma_1^2 = \sigma_2^2 = \tau^2 = [\sigma_1, \tau] = [\sigma_2, \tau] = 1,$$

which can be done by direct matrix computations. This completes the proof that $\mathrm{GLift}(k, n)$ splits in the cases listed above.

In order to complete the proof of Theorem 7.1, it remains to prove that in all other cases $\mathrm{GLift}(k, n)$ is not split.

(1) If $n \geq 3$ and $p \geq 3$, the conclusion follows from Lemma 4.1 and [MS24, Claim 5.4]. (One could replace [MS24, Claim 5.4] by the stronger Theorem 2.5.)

(2) If $p = 2$ and $n \geq 2$ and $|k| > 2$, see Lemma 5.1.

(3) If $p = 2$ and $n \geq 4$, see Proposition 6.11.

(4) If $p > 3$ and $n = 2$, see [MS24, Remark 5.8(1)].

(5) It remains to consider the case $p = 3$, $n = 2$ and $|k| > 3$. Choose $x, y \in k$ which are linearly independent over $\mathbb{F}_3$, and let $\rho \coloneqq I + xE_{12}$ and $\mu \coloneqq I + yE_{12}$. Observe that ρ and μ generate a subgroup $H \cong (\mathbb{Z}/3\mathbb{Z})^2$ of $U_2(k)$. We will show that the restriction of $\mathrm{GLift}(k, 2)$ to H is not trivial. Let $\tilde{x} \coloneqq (x, 0)$ and $\tilde{y} \coloneqq (y, 0)$ in $W_2(k)$, so that $\tilde{\rho} \coloneqq I + \tilde{x}E_{12}$ and $\tilde{\mu} \coloneqq I + \tilde{y}E_{12}$ are lifts of ρ and μ to $\mathrm{GL}_2(W_2(k))$, respectively. Observe that $3(z, 0) = (0, z^3) = \iota(z^3)$ for all $z \in k$. Thus

$$\tilde{\rho}^3 = I + \iota(x^3)E_{12}, \qquad \tilde{\mu}^3 = I + \iota(y^3)E_{12}, \qquad [\tilde{\rho}, \tilde{\mu}] = I.$$

Suppose by contradiction that the restriction of $\mathrm{GLift}(k, 2)$ to H splits. Then, by Lemma 2.9, there exist $U = (u_{ij})$ and $V = (v_{ij})$ in $M_2(k)$ such that

$$N_\rho(U) = x^3 E_{12}, \qquad N_\mu(V) = y^3 E_{12}, \qquad (\rho - 1)V - (\mu - 1)U = 0.$$

On the other hand, a matrix computation shows that $N_\rho(U) = u_{21} x^6 E_{12}$ and $N_\mu(V) = v_{21} y^6 E_{12}$, and that the $(1,1)$-th entry of $(\rho - 1)V - (\mu - 1)U$ is equal to $x^3 v_{21} - y^3 u_{21}$. We obtain $u_{21} = x^{-3}$ and $v_{21} = y^{-3}$, and hence $x^6 = y^6$, that is, $x = \pm y$. This contradicts the fact that x and y are linearly independent over $\mathbb{F}_3$. We conclude that the restriction of $\mathrm{GLift}(k, 2)$ to H does not split, as desired. □

Remark 7.2. In cases (ii)-(iv) of the proof of Theorem 7.1, where $k = \mathbb{F}_p$ for $p \in \{2, 3\}$, the splittings $U_n(\mathbb{F}_p) \to \mathrm{GL}_n(\mathbb{F}_p)$ are integral, that is, they lift to homomorphisms $U_n(\mathbb{F}_p) \to \mathrm{GL}_n(\mathbb{Z})$ defined by the same matrices, this time viewed as matrices with integer coefficients.

References

[Böc03] Gebhard Böckle. Lifting mod p representations to characteristics p^2. *J. Number Theory*, 101(2):310–337, 2003. 2

[Bou06] Nicolas Bourbaki. *Éléments de mathématique. Algèbre commutative. Chapitres 8 et 9*. Reprint of the 1983 original. Springer, Berlin, 2006. 7

[CDF24] Andrea Conti, Cyril Demarche, and Mathieu Florence. Lifting Galois representations via Kummer flags. *arXiv:2403.08888*, 2024. 2

[DCF22] Charles De Clercq and Mathieu Florence. Lifting low-dimensional local systems. *Math. Z.*, 300(1):125–138, 2022. 2, 8, 10

[Flo20] Mathieu Florence. Smooth profinite groups, II: the uplifting theorem. *arXiv:2009.11140*, 2020. 2

[Flo24] Mathieu Florence. Triangular representations that do not lift. *Unpublished note*, 2024. 2

[GM22] Matthew Gherman and Alexander Merkurjev. Negligible degree two cohomology of finite groups. *J. Algebra*, 611:82–93, 2022. 6, 35

[GMS03] Skip Garibaldi, Alexander Merkurjev, and Jean-Pierre Serre. *Cohomological invariants in Galois cohomology*, volume 28 of *University Lecture Series*. American Mathematical Society, Providence, RI, 2003. 5

[Kha97] Chandrashekhar Khare. Base change, lifting, and Serre's conjecture. *J. Number Theory*, 63(2):387–395, 1997. 1, 10

[Kha06] Chandrashekhar Khare. Serre's modularity conjecture: the level one case. *Duke Math. J.*, 134(3):557–589, 2006. 1

[KW09a] Chandrashekhar Khare and Jean-Pierre Wintenberger. Serre's modularity conjecture. I. *Invent. Math.*, 178(3):485–504, 2009. 1

[KW09b] Chandrashekhar Khare and Jean-Pierre Wintenberger. Serre's modularity conjecture. II. *Invent. Math.*, 178(3):505–586, 2009. 1

[KL20] Chandrashekhar Khare and Michael Larsen. Liftable groups, negligible cohomology and Heisenberg representations. *arXiv preprint arXiv:2009.01301*, 2020. 1, 2

[KMRT98] Max-Albert Knus, Alexander Merkurjev, Markus Rost, and Jean-Pierre Tignol. *The book of involutions*, volume 44 of *American Mathematical Society Colloquium Publications*. American Mathematical Society, Providence, RI, 1998. With a preface in French by J. Tits. 11

[MT11] Gunter Malle and Donna Testerman. *Linear algebraic groups and finite groups of Lie type*. Cambridge Studies in Advanced Mathematics, volume 133. Cambridge University Press, Cambridge, 2011. 17

[MS24] Alexander Merkurjev and Federico Scavia. Galois representations modulo p that do not lift modulo p^2. *arXiv preprint arXiv:2410.12560*, 2024. 2, 3, 6, 8, 36

[NSW08] Jürgen Neukirch, Alexander Schmidt, and Kay Wingberg. *Cohomology of number fields*, volume 323 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, second edition, 2008. 5, 7, 11, 16

[Ram99] Ravi Ramakrishna. Lifting Galois representations. *Invent. Math.*, 138(3):537–562, 1999. 1

[Ser91] Jean-Pierre Serre. Quelques problèmes de cohomologie galoisienne, 1991. Cours de Jean-Pierre Serre, no. 12, E. Bayer, Goldstein, C. (red.), 234 p. 5

[Ser94] Jean-Pierre Serre. Quelques exemples d'invariants cohomologiques, 1994. Cours de Jean-Pierre Serre, no. 15 (1993–1994), A. Quéguiner (red.), 178 p. 5

[Ser02] Jean-Pierre Serre. *Galois cohomology*. Springer Monographs in Mathematics. Springer-Verlag, Berlin, 2002. Translated from the French by Patrick Ion and revised by the author. Corrected reprint of the 1997 English edition. 4, 10, 11

[Ser12] Jean-Pierre Serre. *Linear representations of finite groups*, volume 42. Springer Science & Business Media, 2012. 8

[Wei94] Charles A. Weibel. *An introduction to homological algebra*. Cambridge Studies in Advanced Mathematics, volume 38. Cambridge University Press, Cambridge, 1994. 7

[Wil95] Andrew Wiles. Modular elliptic curves and Fermat's last theorem. *Ann. of Math. (2)*, 141(3):443–551, 1995. 1

Department of Mathematics, University of California, Los Angeles, CA 90095, United States of America

Email address: merkurev@math.ucla.edu

CNRS, Institut Galilée, Université Sorbonne Paris Nord, 99 avenue Jean-Baptiste Clément, 93430, Villetaneuse, France

Email address: scavia@math.univ-paris13.fr