A SHORT PROOF THAT AN ACYCLIC n-REGULAR BOREL GRAPH MAY HAVE BOREL CHROMATIC NUMBER $n+1$

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In [2] we show that for every n, there exists an acyclic n-regular Borel graph with Borel chromatic number equal to $n+1$. Our proof in that paper follows from a very general lemma which we use to prove a number of other results. In this short note, we give an easier proof of the aforementioned theorem by streamlining our argument down to only what is needed for this single application.

Theorem. *For every n, there is an acyclic n-regular Borel graph with no Borel n-coloring.*

Proof. Fix an n. Now consider the group $\mathbb{Z}_2^{*n} = \langle \gamma_0, \gamma_1, \ldots, \gamma_{n-1} | \gamma_i^2 = e\rangle$. The Caley graph of $\mathbb{Z}_2^{*n}$ (with respect to the generators $\gamma_1, \gamma_2, \ldots, \gamma_n$) is an n-regular graph. Let $\mathbb{Z}_2^{*n}$ act by left shift on the space $\omega^{\mathbb{Z}_2^{*n}}$. Let X be the set of $x \in \omega^{\mathbb{Z}_2^{*n}}$ such that for all $\alpha \in \mathbb{Z}_2^{*n}$ and all $i < n$, we have $x(\alpha) \neq x(\alpha\gamma_i)$. Note that $\gamma_i \cdot x \neq x$ for all $x \in X$. Now consider the graph G on X where there is an edge between $x, y \in X$ if $\gamma_i \cdot x = y$ for some $i < n$. Note that G is not acyclic.

We will begin by showing that G does not have a Borel n-coloring. To finish, we will show this is true even of the restriction of this graph to the free part of the action of $\mathbb{Z}_2^{*n}$ on X. Let $c : X \to n$ be a Borel function. We will find an x and an $i < n$ such that $c(x) = i = c(\gamma_i \cdot x)$.

For each $i \in n$ and each $j \in \omega$, consider the following game $G_{i,j}$ for building an $x \in X$ where $x(e) = j$. On player I's nth turn, they must define x on the set of reduced words of length n beginning with γ_i. On player's II's nth turn, they must define x on the set of reduced words of length n that do not begin with γ_i. Both players must ensure that $x(g) \neq x(g\gamma_i)$ for all $g \in \mathbb{Z}_2^{*n}$ and $i < n$. Finally, player I wins the game if and only $c(x) \neq i$.

Now it is clear that given any $j \in \omega$, player I can not win the games $G_{i,j}$ for all $i \in n$; we could use winning strategies for player I in all of these games to produce an x that is simultaneously an outcome of all these strategies, and hence $c(x) \neq i$ for all $i \in n$. Thus, for each j, there must be an $i \in n$ such that player II wins $G_{i,j}$. By the pigeonhole principle, there must therefore be an $i \in n$ and distinct $j_0, j_1 \in \omega$ such that player II wins G_{i,j_0} and G_{i,j_1}. In the obvious way, we can use these two winning strategies for player II to produce an x such that $x(e) = j_0$, $x(\gamma_i) = j_1$, x is an outcome of player II's winning strategy in G_{i,j_0} and $\gamma_i \cdot x$ is an outcome of player II's winning strategy in G_{i,j_1}. We have therefore found an x such that $c(x) = i = c(\gamma_i \cdot x)$.

Let $Y \subseteq X$ be the free part of the shift action of $\mathbb{Z}_2^{*n}$ on X. Clearly $G \restriction Y$ is n-regular and acyclic. We will now show that there is a Borel function $c^* : X \setminus Y \to n$ such that for all x and $i \in n$, either $c^*(x) \neq i$, or $c^*(\gamma_i \cdot x) \neq i$. This will complete our proof; given any Borel function $d : Y \to n$, let $c = c^* \cup d$, and apply the argument above to conclude d is not a Borel n-coloring of $G \restriction Y$.

Our construction of c^* will be a special case of [2, Lemma 2.3]. Consider the sequences $\langle x_0, x_1, \ldots, x_{n+1}\rangle$ of elements of $X \setminus Y$ such that $x_0 = x_{n+1}$ and $x_i \neq x_j$ for $i < j \leq n$, and there exists an associated sequence $\langle \gamma_{k_0}, \gamma_{k_1}, \ldots, \gamma_{k_n}\rangle$ such that $\gamma_{k_i} \cdot x_i = x_{i+1}$ and $\gamma_{k_i} \neq \gamma_{k_{i+1}}$. Such sequences witness that the action of $\mathbb{Z}_2^{*n}$ is not free on $X \setminus Y$. By [1, Lemma 7.3], let A be a Borel set of such sequences $\langle x_0, x_1, \ldots, x_{n+1}\rangle$ with associated $\langle \gamma_{k_0}, \gamma_{k_1}, \gamma_{k_n}\rangle$ that are pairwise disjoint, and so that A contains at least one element from each equivalence class of the shift action of $\mathbb{Z}_2^{*n}$ on $X \setminus Y$. For the elements of each such sequence, we define $c^*(x_i) = k_i$. Now note that if $c^*(x)$ is defined and $c^*(x) = i$, then $c^*(\gamma_i \cdot x)$ is also defined and $c^*(\gamma_i \cdot x \neq i)$.

We will finish the construction of c^* in ω many steps while keeping this previous sentence true at every step of the remainder of our construction. Let $i_0, i_1, \ldots$ be a sequence of elements of n containing each number in n infinitely many times. At step m, for all $x \in X \setminus Y$, if $c^*(x)$ is defined and $c^*(\gamma_{i_m} \cdot x)$ is not defined, then set $c^*(\gamma_{i_m} \cdot x) = i_m$. $\square$

References

[1] Alexander S. Kechris and Benjamin D. Miller, *Topics in orbit equivalence*, Lecture Notes in Mathematics, vol. 1852, Springer-Verlag, Berlin, 2004. MR2095154 (2005f:37010)

[2] Andrew Marks, *A determinacy approach to Borel combinatorics* (2013), `arXiv:1304.3830`.

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