**Math 273**: **Homework #3, due on Monday, November 8**

**[1]** Let $\Omega$ be an open and bounded domain in $\mathbb{R}^2$, with sufficiently smooth boundary $\partial\Omega$. Consider the minimization problem in two dimensions

$$\inf_u F(u) = \int_\Omega (Ku - u_0)^2 dxdy + \alpha \int_\Omega f(\nabla u) dxdy,$$

with $u_0 \in L^2(\Omega)$ (square integrable function) a given function, and $f$ is a smooth function on $\mathbb{R}^2$ with real values. Here $K : L^2(\Omega) \to L^2(\Omega)$ is a linear and continuous operator, and its adjoint is $K^*$ (thus $K^*$ has the property $\int_\Omega (Ku)v dxdy = \int_\Omega u(K^*v) dxdy$). Obtain, as in Hw#1[5] and Hw#2[1], the Euler-Lagrange equation associated with the minimization problem that is formally satisfied by a sufficiently smooth optimal $u$. No explicit boundary conditions are imposed, thus you have to deduce implicit (or natural) boundary conditions on $\partial\Omega$.

**[2]** (This problem is related with [5] from Hw #1)

Let $u(x, y, t)$ be a smooth solution of the time-dependent PDE

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x} L_{u_x}(P) + \frac{\partial}{\partial y} L_{u_y}(P) - L_u(P),$$

with $u(x, y, 0) = u_0(x, y)$ in $\Omega$ and $u(x, y, t) = g(x, y)$ for $(x, y) \in \partial\Omega$ and $t \geq 0$ (recall that $P$ is a notation for $(x, y, u(x, y), u_x(x, y), u_y(x, y))$).

Show that the function $E(t) = F(u(\cdot, \cdot, t))$ is decreasing in time, where $F(u) = \int_\Omega L(x, y, u, u_x, u_y) dxdy$.

**[3]** Apply the gradient descent method described in class to the two-dimensional diffusion problem

$$F(u) = \sum_{1 \leq i,j \leq n} \left[ (u_{i+1,j} - u_{i,j})^2 + (u_{i,j+1} - u_{i,j})^2 + \lambda (u_{i,j} - f_{i,j})^2 \right],$$

where $f_{i,j}$ is given for $0 \leq i, j \leq n+1$, and with boundary conditions $u_{i,j} = f_{i,j}$ if $i = 0$ or $i = n+1$ or $j = 0$ or $j = n+1$ (chosen for simplicity). Here $\lambda > 0$ is a tunning parameter. Choose a function $f$ of your choice (for example an image). If you do not have one, you can create a synthetic image. Test various values of the parameter $\lambda$ and observe the properties of your implementation. Give your choice of the stopping criterion and also plot the value of the objective function versus steps. Plot the data $f$, your starting point and your final result, as 2D images.

**[4]** Consider the constrained optimization problem

$$\min_{x \in \mathbb{R}^n} f(x), \quad \text{subject to } Ax = b,$$

where $A \in \mathbb{R}^{m \times n}$ has full row rank, $m \leq n$, $b \in \mathbb{R}^m$. Tranform the problem into an unconstrained minimization problem by one of the methods discussed in class.

**[5]** Verify that the KKT conditions (1st order optimality conditions) for the bound-constrained problem

$$\min_{x \in \mathbb{R}^n} \phi(x), \text{ subject to } l \leq x \leq u,$$

are equivalent to the compactly stated condition $P_{[l,u]} \nabla \phi(x) = 0$, where the projection operator $P_{[l,u]}$ of a vector $g \in \mathbb{R}^n$ onto the rectangular box $[l, u]$ is defined by

$$(P_{[l,u]} g)_i = \begin{cases} min(0, g_i), & \text{if } x_i = l_i, \\ g_i, & \text{if } x_i \in (l_i, u_i), \text{ for all } i = 1, 2, ..., n \\ max(0, g_i), & \text{if } x_i = u_i. \end{cases}$$