Math 273:
Homework #5: tentative due date Friday, December 4
(7 problems)
(You can also return this last homework during the week of finals, but no later than December 10. You can leave it in my mailbox in MS 6363, at my office, or with Babette Dalton at MS 7619).

[1] Assume that V and V^* are normed vector spaces in duality. Let $F : V \to \overline{\mathbb{R}}$ and let $F^* : V^* \to \overline{\mathbb{R}}$ be the polar or conjugate of F. We define F^* by

$$F^*(u^*) = \sup_{u \in V} \Big\{ \langle u^*, u \rangle - F(u) \Big\}.$$

Show
(i) $F^*(0) = -inf_{u\in V} F(u)$.
(ii) $(\lambda F)^*(u^*) = \lambda F^*(\frac{1}{\lambda} u^*)$ for every $\lambda > 0$.

[2] Let $V = V^* = \mathbb{R}^n$. Let Q be a symmetric positive definite $n \times n$ matrix, $b \in \mathbb{R}^n$, and consider $f(x) := \frac{1}{2}\langle x, Qx\rangle + \langle b, x\rangle$, for all $x \in \mathbb{R}^n$. Find the polar (or the conjugate) f^* and deduce that, in particular, $\frac{1}{2}\|\cdot\|^2$ is its own polar (or conjugate).

[3] Let $F : V \to \mathbb{R}$ and F^* its polar. Then $u^* \in \partial F(u)$ if and only if $F(u) + F^*(u^*) = \langle u^*, u\rangle$.

[4] Show that the polar F^* is convex.

[5] Let $F : \mathbb{R}^n \to \mathbb{R}$ be continuous and coercive with respect to the Euclidean norm $\|x\|$, for $x \in \mathbb{R}^n$. Then there is $x_0 \in \mathbb{R}^n$ such that $F(x_0) = \inf_{x\in\mathbb{R}^n} F(x)$. If in addition F is strictly convex, then the minimizer x_0 is unique.

[6] Let $f \in \mathbb{R}^{N^2}$ be given, and let $u \in \mathbb{R}^{N^2}$ be an unknown minimizer of the functional (already seen before)

$$E(w) = \sum_{i,j=0}^{N-1} |\nabla w_{i,j}|^2 + \lambda \sum_{i,j=0}^{N-1} (w_{i,j} - f_{i,j})^2,$$

for $w \in \mathbb{R}^{N^2}$, where

$$\nabla w_{i,j} = \begin{pmatrix} (D_x w)_{i,j} \\ (D_y w)_{i,j} \end{pmatrix} = \begin{pmatrix} w_{i+1,j} - w_{i,j} \\ w_{i,j+1} - w_{i,j} \end{pmatrix},$$

for $(i, j) \in \{0, ..., N-1\}^2$ (we assume that all vectors f, u, w are periodic outside of their support).

(a) Find the adjoint operators D_x^* and D_y^* of D_x and D_y.

(b) Find a linear operator $B : \mathbb{R}^{N^2} \to \mathbb{R}^{N^2}$, a $c \in \mathbb{R}^{N^2}$, and $C(f)$, independent of w, such that for all $w \in \mathbb{R}^{N^2}$,

$$E(w) = \langle Bw, w\rangle + \langle c, w\rangle + C(f).$$

(c) Show that B is self-adjoint.

(d) Find the Gateaux differential of $E(w)$ in the direction v and thus give a necessary (and sufficient) condition for u to be a minimizer, by setting this differential to zero (as the zero functional).

[7](*Rudin-Osher-Fatemi Edge-preserving denoising application in image processing*) Numerically solve by any method of your choice (from the lecture or not necessarily from the lecture) the minimization of

$$F(u) = \sum_{1\leq i,j\leq n} \left[\sqrt{1 + (u_{i+1,j} - u_{i,j})^2 + (u_{i,j+1} - u_{i,j})^2} + \lambda(u_{i,j} - f_{i,j})^2\right],$$

where $f_{i,j}$ is given for $1 \leq i, j \leq n$, and with Neumann boundary conditions in u. Here $\lambda > 0$ is a tunning parameter. (uou can add noise to a clean image using the matlab command "imnoise").

Give the details of your method. Test various values of the parameter λ and observe the properties of your implementation; choose λ that visually will give you a satisfactory result (noise is removed, while restored image u is sharp). Give your choice of the stopping criteria and also plot the value of the objective function versus steps. Plot the data f, your starting point and your final result, as 2D images (you can use your starting point the noisy image f, or a constant). Also, give the details of your numerical implementation and include your code. If you need a clean image and a noisy version f that can be opened in matlab, you can send me an e-mail.

Optional: If someone would like to solve a different optimization computational exercise instead of the one above, from your own research, you are welcome to do so. In this case, include more details of the problem.