Math 269B Instructor: Luminita Vese. Teaching Assistant: Michael Puthawala.
Homework #6 Due on: Friday, March 3rd, or exceptionally the following Monday.

[**1**] Consider the one-way wave equation $u_t + u_x = 0$ for $t > 0$ and $x \in R$, with the initial condition $u(x, 0) = u_0(x)$.

(a) Give the exact solution of the equation.

(b) Show that the leapfrog scheme with $\lambda = \frac{k}{h} = 1$ applied to this equation gives the exact solution (let v_m^0 and v_m^1 be given by the exact solution at $t = 0$ and $t = k$, for all $m \in Z$).

[**2**] Consider the equation

$$u_t = b_{11}u_{xx} + 2b_{12}u_{xy} + b_{22}u_{yy}.$$

(a) Under what conditions is this a parabolic equation ?

(b) Derive a formula for the solution of this equation using the Fourier transform in two spatial dimensions, function of $\hat{u}_0(\omega_1, \omega_2)$.

[**3**] Consider the convection-diffusion equation with lower order term,

$$u_t + au_x = bu_{xx} + cu,$$

with $b > 0$ and all constant coefficients, and with initial condition $u(0, x) = u_0(x)$, for x real.

(a) Let $y = x - at$ and set $w(t, y) = u(t, y + at)$. Find an equation satisfied by w.

(b) Derive a formula for the solution of the equation in u using the Fourier transform.

(c) Show that the (IVP) in u is well-posed (in the sense of the Definition 1.5.2 from the textbook by J. Strikwerda).

[**4**] Show that the two-dimensional Du Fort-Frankel scheme for the equation $u_t = b(u_{xx} + u_{yy}) + f$ given by

$$\frac{v_{l,m}^{n+1} - v_{l,m}^{n-1}}{2k} = b\frac{v_{l+1,m}^n + v_{l-1,m}^n + v_{l,m+1}^n + v_{l,m-1}^n - 2(v_{l,m}^{n+1} + v_{l,m}^{n-1})}{h^2} + f_m^n,$$

where $\triangle x = \triangle y = h$, $b > 0$, is unconditionally stable.

[**5**] Consider the system of partial differential equations

$$(1) \qquad \begin{cases} u_t + u_x + v_x = 0, & u(x,0) = u_0(x), \\ v_t + u_x - v_x = 0, & v(x,0) = v_0(x). \end{cases}$$

(a) Write the system in matrix form $U_t + AU_x = \vec{0}$, with $U(x,t) = \begin{pmatrix} u(x,t) \\ v(x,t) \end{pmatrix}$, and find the matrix A.

(b) Is this a hyperbolic system ? Explain.

(c) Reduce, if possible, the system (1) to a set of independent scalar hyperbolic equations.

(d) Obtain the exact solution of the system (1).

(e) Let $V_m^n \approx \begin{pmatrix} u(x_m, t_n) \\ v(x_m, t_n) \end{pmatrix}$. Consider the FTCS scheme applied to the system (1) in matrix form:

$$(2) \qquad \frac{1}{k}(V_m^{n+1} - V_m^n) + \frac{1}{2h}A(V_{m+1}^n - V_{m-1}^n) = \vec{0}.$$

Find the nonsingular amplification matrix G for this scheme by substituting $V_m^n = G^n e^{im\theta}$ into the difference equation (2).

(f) Apply the statements below to the particular matrices A and G from (a)-(e) and analyze the stability of the scheme when $\lambda = \frac{k}{h}$ is constant.

• We have:

Assume that A is a diagonalizable matrix (i.e. there is a nonsingular matrix P such that $PAP^{-1} = \Lambda_A$ is diagonal). If $G = \Phi(A)$, where $\Phi(A) = c_0 I + c_1 A + c_2 A^2$ is a polynomial of A, whith I the identity matrix, and c_0, c_1, c_2 constants, then G is also diagonalizable, with the same matrix P.

Moreover, the diagonal matrix Λ_G of eigenvalues of G can easily be obtained function of Λ_A (in fact $\Lambda_G = \Phi(\Lambda_A)$).

Note 1. The above statement is true for any polynomial or rational function Φ of A.

Note 2. Assume that A and G are as in the general statement, with A a constant matrix, and G the amplification matrix. Then the von Neumann condition is necessary and sufficient for stability (hint: write $G = P^{-1}\Lambda_G P$, $\|G^n\|_2 = \|(P^{-1}\Lambda_G P)^n\|_2 \leq \|P^{-1}\|_2 \|P\|_2 \|\Lambda_G^n\|_2$, and use the general formula $\|M\|_2 = \rho(M^*M)$, where M is an nxn matrix and $\rho(M^*M)$ is the spectral radius of M^*M).

Recall the Von Neumann stability condition: $\forall\ T > 0$, there is $C_T > 0$ s.t. for $0 \leq nk \leq T$ we have: $\|G^n\| \leq C_T$.