Midterm: Thursday, February 17, 3-4pm.

Topics covered for the midterm: stability, consistency and order of accuracy for finite-difference schemes for the one-way wave equation and the diffusion equation in one space dimension. Convergence.

- Stability and consistency for schemes for systems are NOT covered for the midterm.
- Stability by von Neumann analysis for multi-step schemes is NOT covered for the midterm.

Practice problems for the midterm (you should first review the lecture notes and the homework assignments)

[1] Consider the system

$$\begin{cases} u_t + 2u_x + v_x = 0 \\ v_t + u_x + 2v_x = 0 \end{cases},$$

with initial data $u_0(x) = u(x,0) = \begin{cases} 1 \text{ if } |x| \le 1 \\ 0 \text{ if } |x| > 1 \end{cases}$, $v(x,0) = v_0(x) = 0$.

(a) Write the system in matrix-vector form.

(b) Find the exact solution of the system. Hint: transform the system into a set of two uncoupled one-way wave equations.

[2] Consider the one-way wave equation $u_t + au_x = 0$ for $t > 0$ and $x \in R$, with the initial condition $u(x,0) = u_0(x)$.

(a) Give the exact solution of the equation.

(b) Show that the leapfrog scheme (C-T,C-S) is consistent with the one-way wave equation $u_t + au_x = 0$.

(c) Show that the leapfrog scheme with $\lambda = \frac{\triangle t}{h} = 1$ applied to $u_t + u_x = 0$ gives the exact solution (let u_j^0 and u_j^1 be given by the exact solution at $t = 0$ and $t = \triangle t$, for all $j \in Z$).

[3] Show that the following scheme is consistent with the one-way wave equation $u_t + au_x = f(x,t)$:

$$\frac{u_j^{n+1} - u_j^n}{\triangle t} + \frac{a}{2}\Big(\frac{u_{j+1}^{n+1} - u_j^{n+1}}{h} + \frac{u_j^n - u_{j-1}^n}{h}\Big) = f_j^n.$$

[4] Using L^2 norm stability (not Von Neumann analysis) find and prove a sufficient condition of stability for schemes of the form

$$u_j^{n+1} = \alpha u_j^n + \beta u_{j-1}^n.$$

Apply your result to the forward-time backward space scheme for the one-way wave equation $u_t + au_x = 0$.

[5] Consider the modified Lax-Friedrichs scheme for the one-way wave equation $u_t + au_x = f(x,t)$,

$$u_j^{n+1} = \frac{1}{2}(u_{j+1}^n + u_{j-1}^n) - \frac{a\lambda}{1+(a\lambda)^2}(u_{j+1}^n - u_{j-1}^n) + \triangle t f_j^n.$$

(a) Analize the consistency and the order of the scheme.

(b) Show that this explict scheme is stable for all values of λ. Discuss the relation of this explicit and unconditionally stable scheme with the theorem about the Courant-Friedrichs-Lewy condition.

[6] Show that the backward-time central-space scheme is consistent with the one-way wave equation and is uncoditionally stable (use von Neumann analysis).

[7] Using von Neumann analysis, show that the reverse Lax-Friedrichs scheme from Assignment 4 is stable for $|a\lambda| \geq 1$.

[8] Show that the scheme forward-time central-space for $u_t + au_x = bu_{xx}$ satisfies the condition $|g| \leq 1$ if and only if $\triangle t \leq 2b/a^2$.