Local regularization of the One-Phase Hele-Shaw Flow

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Abstract

This article presents a local regularity theorem for the one-phase Hele-Shaw flow. We prove that if the Lipschitz constant of the initial free boundary in a unit ball is small, then for small uniform positive time the solution is smooth. This result improves on our earlier results in [CJK] because it is scale-invariant. As a consequence we obtain existence, uniqueness and regularity properties of global solutions with Lipschitz initial free boundary.

1   Introduction

For $(x,t) \in \mathbb{R}^n \times \mathbb{R}$ (or some subset) let $u(x,t)$ be a viscosity solution (defined in the next section) of the Hele-Shaw problem

$$(HS) \qquad \begin{cases} -\Delta u(\cdot, t) = 0 & \text{in } \{u > 0\}, \\[2mm] \frac{\partial u}{\partial t} = |Du|^2 & \text{on } \partial\{u > 0\} \end{cases}$$

where Du denotes the gradient of u in the space variables x. We refer to

$$\Omega_t(u) = \{x : u(x,t) > 0\} \subset \mathbb{R}^n, \quad \Omega(u) = \{(x,t) : u > 0\} \subset \mathbb{R}^{n+1}$$
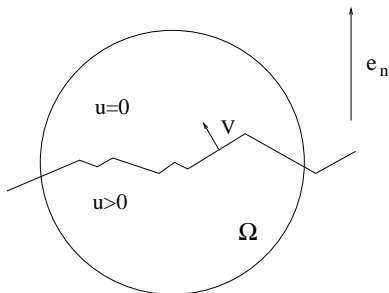
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Figure 1: Initial setting of the problem

as the *positive phase* of u and to

$$\Gamma_t(u) = \partial\Omega_t(u), \quad \Gamma(u) = \partial\Omega(u)$$

as the *free boundary* of u.

If u is smooth up to the free boundary, then the free boundary moves with normal velocity $V = u_t/|Du|$, and hence the second equation in (HS) implies that

$$V = |Du| = -\frac{\partial u}{\partial \nu},$$

where ν is the outward spatial normal, with respect to $\Omega_t(u)$, on $\Gamma_t(u)$. (see Figure 1)

The classical Hele-Shaw problem models an incompressible viscous fluid which occupies part of the space between two parallel, nearby plates, and u is the pressure of the fluid. The short-time existence of classical solutions with initial surface Γ_0 of class $C^{2+\alpha}$ was proved by Escher and Simonett [ES]. When $n = 2$, Elliot and Janovsky [EJ] showed the existence and uniqueness of weak solutions formulated by a parabolic variational inequality in a Sobolev space. (See also Gustafsson [G].) Using this variational notion of solutions, Sakai [S] showed that for $n = 2$, $\Gamma_t(u)$ for all time $t > 0$ is either analytic or has a cusp-type singularity caused by collision of two free boundary components.

In this paper we prove a quantitative, geometric version of Sakai's theorem in general dimensions. Of independent interest is an estimate on the free boundary speed, or, equivalently, the size of the spatial gradient on the free boundary (part (c) of Theorems 1.1 and 1.2).

Our first main theorem concerns *global solutions* starting from a Lipschitz initial free boundary. (See section 9 for the precise definition of global

solutions.) Let $e_n = (0, \ldots, 0, 1) \in I\!\!R^n$ and

$$\Omega_0 = \{x \in I\!\!R^n : x_n < g(x'), x = (x', x_n) \in I\!\!R^{n-1} \times I\!\!R\}$$

where g is Lipschitz continuous with Lipschitz constant $L \leq L_n$, a dimensional constant. There is a positive harmonic function u_0 in Ω_0 that vanishes on $\partial\Omega_0$, and it is unique up to a constant multiple. Because there is only uniqueness up to a constant multiple, we must normalize the solution at subsequent times t.

Theorem 1.1. *(Theorem 9.2) There exists a unique, global solution u of (HS) in $I\!\!R^n \times [0, \infty)$ with initial positive phase $\Omega_0(u) = \Omega_0$, $u(x, 0) = u_0(x)$, normalized at infinity by*

$$\lim_{R \to \infty} \frac{u(-Re_n, t)}{u_0(-Re_n)} = 1 \text{ for each fixed } t > 0.$$

Moreover the following properties hold.

  (a) For $t > 0$ the free boundary $\Gamma_t(u)$ is smooth and is a Lipschitz graph with respect to e_n with Lipschitz constant L;

  (b) u is smooth in $\bar{\Omega}_t(u)$ for $t > 0$;

  (c) If $x \in \Gamma_0(u)$ and $x + de_n \in \Gamma_t(u)$ with $d \geq 0$, then

$$C^{-1} \leq \frac{|Du(x + de_n, t)|}{|Du(x - de_n, t)|} \leq C$$

  for a constant $C > 0$ depending only on dimension.

Note that part (a) says that the Lipschitz constant is preserved (or decreases) with time. In fact, we show in Theorem 9.2(d) that if $g(-x') = g(x')$, then $u(-x', x_n, t) = u(x', x_n, t)$, and the Lipschitz constant decreases to zero locally uniformly as $t \to \infty$.

L_n can be taken to be any number $L_n < a_n$, where a_n is the Lipschitz constant appeared in our earlier work [CJK]. For $n = 2$ we have $a_2 = 1$, for which the initial harmonic function $u(x, 0)$ vanishes on the boundary at a rate bounded below by the square of the distance to the boundary. For a Lipschitz constant $L < a_2$, some points of the free boundary can stay fixed

for small $t > 0$, and the free boundary need not be smooth for those values of t. (See [JK], [KLV].)

Theorem 1.1 is proved using Theorem 1.2, our main local theorem. We also show that blow-up limits of local solutions are global solutions as in Theorem 1.1. The local theorem says, roughly speaking, that u and $\Gamma_t(u)$ stay smooth as long as there is no invasion of fingers (another component of $\Omega(u)$) from the top portion of the local neighborhood.

Let Ω_0 be, as above, the region below the graph of the Lipschitz function g with Lipschitz constant $L \leq L_n$. Assume further that $g(0) = 0$.

Theorem 1.2. *(Corollary 5.6, Theorem 6.3 and Corollary 7.4) Suppose u is a solution of (HS) in $B_2(0) \times [0, T]$ with initial positive phase $\Omega_0 \cap B_2(0)$, and u satisfies $u(-e_n, 0) = 1$ with*

$$u(x, s) \leq 10u(x, t), \quad u(x, s) \leq (1 + At^{-1+a}(t - s))u(x, t) \text{ for } 0 \leq s < t \leq T,$$
$$\tag{1.1}$$
$$u(-e_n, t) \leq Au(-e_n, 0) = A \text{ for } 0 \leq t \leq T, \tag{1.2}$$

and
$$\Omega_T(u) \cap B_2(0) \subset (\Omega_0 + c_0 e_n). \tag{1.3}$$

Then there is a constant $c_0 > 0$ depending only on dimension and on $A > 0$ such that the following properties hold:

If $0 < r < c_0$ is chosen sufficiently small that $u(re_n, T/2) > 0$, then

(a) *The free boundary $\Gamma_t(u)$ is C^1 and is a Lipschitz graph with respect to e_n with Lipschitz constant $L' < L_n$ in $B_r(0)$, and $L' \to L$ as $r \to 0$.*

(b) *The spatial normal of $\Gamma_t(u)$ in $B_r(0)$ is continuous in space and time.*

(c) *If $x \in \Gamma_0(u) \cap B_r(0)$ and $x + de_n \in \Gamma_t(u) \cap B_r(0)$, then*

$$C^{-1} \leq \frac{|Du(x + de_n, t)|}{|Du(x - de_n, t)|} \leq C$$

for a constant $C > 0$ depending only on dimension.

The role of constant A in Theorem 1.2 is clarified in the body of the paper, where a slightly stronger theorem is proved than is stated here. Our previous theorem in [CJK] gives smoothness for sufficiently small time, but

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depends on global features of the initial configuration. That theorem also treated only constant boundary data $f(x,t) = c$.

Concerning the validity of the hypotheses, we show in Lemma 2.14 that (1.1) is satisfied by suitably re-scaled solutions of (HS) in $(I\!R^n - K) \times [0, \infty)$ with $u(x,t) = f(x,t)$, $x \in K$ for a fixed boundary K, provided $f(x,t)$ is smooth and is strictly positive. Hypothesis (1.3) serves to define the time scale T: it is chosen so that the free boundary moves at most a small multiple of unit distance in time T. The distance r is the distance the free boundary moves in a comparable time $T/2$. The normalization $u(-e_n, 0) = 1$ and the second condition in (1.1) implies that $T \leq T_0(n, A)$. On the other hand, due to (1.2) T can be chosen to be bounded below by a positive constant depending only on the dimension. Thus under conditions (1.1) and (1.2), one should think of T as comparable to unit size.

Hypothesis (1.2) is the last vestige of a global hypothesis. It is valid if T is sufficiently small and in many typical situations. Because it measures the change in size of u at a point well inside the region, it can only fail if there is a change that swamps the effects of the initial conditions. We believe that (1.2) only goes wrong when there are collisions.

The main idea of the proof is to compare solutions at each scale to solutions on star-shaped Lipschitz domains, to which the estimates of [CJK] apply. We use the notion of viscosity solutions and the main tool is the comparison principle (Theorem 2.7). The crucial first step is a suitably localized Carleson-type estimate.

In section 2 we state several preliminary results including the definition and properties of viscosity solutions of (HS). In section 3 we prove a Carleson-type estimate (3.1). This property is used in Section 4 to show that level sets of u are flat (Proposition 4.1). In Section 5 we prove the non-degeneracy of the solution on the free boundary and the Lipschitz continuity of $\Gamma(u)$. In Section 6 and 7 iteration methods developed in [CJK] are used to derive further spatial regularity of the solution and the free boundary. The key steps in the iteration process are estimates on the change of direction of Du over time (Lemma 6.1 and Lemma 7.1). In Section 8 we prove regularity of $\Gamma(u)$ in time. In Section 9, as an application of the main results obtained in previous sections, we prove existence, uniqueness and regularity properties of global solutions of (HS) with Lipschitz initial free boundary with appropriate conditions at infinity. At the same time, we show that the blow-up limits of local solutions are global solutions of the type in Theorem 1.1.

2 Preliminary results

We will recall the definition of *viscosity solutions* for (HS) introduced in [K1]. Roughly speaking, viscosity sub and supersolutions are defined by comparison with local (smooth) super and subsolutions. In particular, classical solutions of (HS) are also viscosity sub and supersolutions.

Let K be a compact subset of $\mathbb{R}^n$ and $Q = (\mathbb{R}^n - K) \times (0, \infty)$. Also let Σ be a cylindrical domain $D \times (a, b) \subset \mathbb{R}^n \times \mathbb{R}$, where D is an open subset of $\mathbb{R}^n$.

For a nonnegative real valued function $u(x, t)$ defined in a cylindrical domain $D \times (a, b)$,

$$\Omega(u) = \{(x, t) : u(x, t) > 0\}, \quad \Omega_t(u) = \{x : u(x, t) > 0\},$$

$$\Gamma(u) = \partial\{(x, t) : u(x, t) = 0\}, \quad \Gamma_t(u) = \partial\{x : u(x, t) = 0\}.$$

Definition 2.1. *A nonnegative upper semi-continuous function u defined in Σ is a* viscosity subsolution *of* (HS) *if*

 (a) for each $a < T < b$ the set $\overline{\Omega(u)} \cap \{t \le T\}$ is bounded; and

 (b) for every $\phi \in C^{2,1}(\Sigma)$ such that $u - \phi$ has a local maximum in $\overline{\Omega(u)} \cap \{t \le t_0\} \cap \Sigma$ at (x_0, t_0),

  (i) $-\Delta\phi(x_0, t_0) \le 0$ *   if $u(x_0, t_0) > 0$.*

  (ii) $(\phi_t - |D\phi|^2)(x_0, t_0) \le 0$ *   if $(x_0, t_0) \in \Gamma(u)$ if $-\Delta\phi(x_0, t_0) > 0$.*

Note that because u is only lower semi-continuous there may be points of $\Gamma(u)$ at which u is positive.

Definition 2.2. *A nonnegative lower semi-continuous function v defined in Σ is a* viscosity supersolution *of* (HS) *if for every $\phi \in C^{2,1}(\Sigma)$ such that $v - \phi$ has a local minimum in $\Sigma \cap \{t \le t_0\}$ at (x_0, t_0),*

$$(i) \qquad -\Delta\phi(x_0, t_0) \ge 0 \qquad if\ v(x_0, t_0) > 0,$$

$$(ii) \qquad If\ (z_0, t_0) \in \Gamma(v), |D\phi|(x_0, t_0) \ne 0\ and$$
$$-\Delta\varphi(x_0, t_0) < 0,$$

 then

$$(\phi_t - |D\phi|^2)(x_0, t_0) \ge 0.$$

Definition 2.3. *u is a viscosity subsolution of (HS) with initial data u_0 and fixed boundary data $f > 0$ if*

(a) u is a viscosity subsolution in $\bar{Q}$,

(b) $u = u_0$ at $t = 0$; $u \leq f$ on ∂K;

(c) $\overline{\Omega(u)} \cap \{t = 0\} = \overline{\Omega(u_0)}$;

Definition 2.4. *u is a viscosity supersolution of (HS) with initial data u_0 and fixed boundary data f if v is a viscosity supersolution in $\bar{Q}$ with $v = v_0$ at $t = 0$ and $v \geq f$ on ∂K.*

For a nonnegative real valued function $u(x, t)$ defined in a cylindrical domain $D \times (a, b)$,

$$u^*(x, t) = \limsup_{(\xi, s) \in D \times (a, b) \to (x, t)} u(\xi, s).$$

Definition 2.5. *u is a viscosity solution of (HS) (with boundary data u_0 and f) if u is a viscosity supersolution and u^* is a viscosity subsolution of (HS) (with boundary data u_0 and f.)*

Definition 2.6. *We say that a pair of functions $u_0, v_0 : \bar{D} \to [0, \infty)$ are (strictly) separated (denoted by $u_0 \prec v_0$) in $D \subset \mathbb{R}^n$ if*

(i) the support of u_0, $\mathrm{supp}(u_0) = \overline{\{u_0 > 0\}}$ restricted in $\bar{D}$ is compact and

(ii) in $\mathrm{supp}(u_0) \cap \bar{D}$ the functions are strictly ordered:

$$u_0(x) < v_0(x).$$

The following properties of viscosity solutions are frequently used in our paper.

Theorem 2.7. *(Comparison principle, [K1]) Let u, v be respectively viscosity sub- and supersolutions in $D \times (0, T) \subset Q$ with initial data $u_0 \prec v_0$ in D. If $u \leq v$ on ∂D and $u < v$ on $\partial D \cap \bar{\Omega}(u)$ for $0 \leq t < T$, then $u(\cdot, t) \prec v(\cdot, t)$ in D for $t \in [0, T)$.*

Theorem 2.8. *([K2]) Let u be a viscosity solution of (HS). Then u is harmonic in $\Omega(u)$. Indeed $u(x, t) = h_t(x)$, where*

$$h_t(x) = \inf\{v \in \mathcal{P} \text{ with } v = 1 \text{ on } \partial K \text{ and } v \geq 0 \text{ on } \Gamma_t\}.$$

where $\mathcal{P}$ is the set of superharmonic functions in Ω_t which are lower semi-continuous in $\bar{\Omega}_t$.

Moreover $\Gamma(u^*) = \Gamma(u)$.

Next we state several properties of harmonic functions:

Lemma 2.9. *(Dahlberg, see [D]) Let u_1, u_2 be two nonnegative harmonic functions in a domain D of $I\!\!R^n$ of the form*

$$D = \{(x', x_n) \in I\!\!R^{n-1} \times I\!\!R : |x'| < 2, |x_n| < 2M, x_n > f(x')\}$$

with f a Lipschitz function with constant less than M and $f(0) = 0$. Assume further that $u_1 = u_2 = 0$ along the graph of f. Then for

$$D_{1/2} = \{|x'| < 1, |x_n| < M, x_n > f(x')\}$$

we have

$$0 < C_1 \leq \frac{u_1(x', x_n)}{u_2(x', x_n)} \cdot \frac{u_2(0, M)}{u_1(0, M)} \leq C_2$$

with C_1, C_2 depending only on M.

Lemma 2.10. *(Jerison and Kenig, see [JK]) Let D, u_1 and u_2 be as in Lemma 2.9. Assume further that*

$$\frac{u_1(0, M/2)}{u_2(0, M/2)} = 1.$$

Then, $u_1(x', x_n)/u_2(x', x_n)$ is Hölder continuous in $\bar{D}_{1/2}$ for some coefficient α, both α and the C^α norm of u_1/u_2 depending only on M.

Lemma 2.11. *(Caffarelli, see [C1]) Let u be as in Lemma 2.9. Then there exists $c > 0$ depending only on M such that for $0 < d < c \frac{\partial}{\partial x_n} u(0, d) \geq 0$ and*

$$C_1 \frac{u(0, d)}{d} \leq \frac{\partial u}{\partial x_n}(0, d) \leq C_2 \frac{u(0, d)}{d}$$

where $C_i = C_i(M)$.

Lemma 2.12. *(Caffarelli, see [C1]) Let u be harmonic in B_1. Then there exists $\epsilon_0 > 0$ such that if*

$$u(x + \epsilon e) \geq u(x) \text{ for } \epsilon > \epsilon_0 \text{ and } x, x + \epsilon e \in B_1(0)$$

for a unit vector $e \in I\!\!R^n$ then $e \cdot \nabla u \geq 0$ in $B_{1/2}(0)$.

Lemma 2.13. *([JeKi], Lemma 4.1) Let Ω be Lipschitz domain contained in $B_{10}(0)$. There exists a dimensional constant $\beta_n > 0$ such that for any $\zeta \in \partial\Omega$, $0 < 2r < 1$ and positive harmonic function u in $\Omega \cap B_{2r}(\zeta)$, if u vanishes continuously on $B_{2r}(\zeta) \cap \partial\Omega$, then for $x \in \Omega \cap B_r(\zeta)$,*

$$u(x) \le C(\frac{|x - \zeta|}{r})^{\beta_n} \sup\{u(y) : y \in \partial B_{2r}(\zeta) \cap \Omega\}$$

where C depends only on the Lipchitz *constants of Ω.*

We finish this section by showing that (1.1) is a generic assumption.

Lemma 2.14. *Let w solve (HS) in $(I\!\!R^n - K) \times [0, \infty)$ with smooth boundary data $f > 0$ on ∂K and initial positive phase Ω_1. Let us normalize such that $u(-e_n, 0) = 1$. Then (1.1) holds with*

$$T = \sup\{t : \sup_{x_i \in K, 0 \le \tau_i \le t} \frac{f(x_1, \tau_1)}{f(x_2, \tau_2)} \le 10\},$$

and with

$$A = \sup\{f_t(x, t) : x \in K, 0 \le t \le T\}.$$

Proof. 1. Let A as given above. Then for any $T > 0$ we have $f(x, t) \le (1 + A\epsilon)f(x, s)$ where $|t - s| \le \epsilon$ and $0 \le t, s \le T$. By Theorem 2.7, for any $\epsilon > 0$

$$u(x, t + t_0) \le (1 + A\epsilon t)u(x, (1 + \epsilon)t + t_0) \text{ for } 0 \le t \le 1/A$$

for any $0 \le t_0$. Differentiating in $\epsilon > 0$ yields

$$(Atu + tu_t)(x, t + t_0) = t(A_0 u + u_t)(x, t + t_0) > 0.$$

Since t_0 is arbitrary, we conclude

$$u_t \ge -Au \text{ for } 0 \le t \le T$$

This proves the second inequality in (1.1).

2. By definition of f, $f \in [a, 10a]$ with $a > 0$ for $0 \le t \le T$. Since the positive phase $\Omega_t(u)$ expands in time, it is clear from the maximum principle of harmonic functions that $u(x, s) \le 10u(x, t)$ if $0 \le s < t \le T$. □

3 A Carleson-type estimate

Our first goal is to show the ϵ-flat result, i.e. to show that the level sets of u stays ϵ-close to those of Lipschitz functions with small Lipschitz constants, at least for a small amount of time. For this we will first prove that u is "almost" increasing in the direction of $-e_n$ in appropriate space-time scale. From this section, suppose u is a solution of (HS) in $B_2(0) \times [0, T]$ with initial positive phase $\Omega_0 \cap B_2(0)$.

Proposition 3.1. *There is a constant $c_0 > 0$ depending only on dimension and on $A > 0$ such that if u satisfies the first condition of (1.1), (1.2) and (1.3), and $0 < r < c_0$ is sufficiently small that $u(re_n, T/2) > 0$, then the following property holds: for small $t > 0$ such that $u(re_n, t) = 0$,*

$$\max_{x \in \partial B_a(0)} u(x, t) \leq C u(-ae_n, t) \tag{3.1}$$

where $a = r^\gamma > r$, $0 < \gamma < 1$ and C are dimensional constants.

$\diamond$ For $\Lambda \subset I\!\!R^n$, denote

$$\Lambda + se_n = \{x + se_n : x \in \Lambda\}.$$

$\diamond$ For $P \in B_2(0) \cap (I\!\!R^n - \bar{\Omega}_0)$, denote by $t(P; u)$, the time the free boundary of u reaches P, i.e.,

$$t(P; u) = \sup\{t > 0 : u(P, t) = 0\}.$$

The proof of Proposition 3.1 follows from the following lemma:

Lemma 3.2. *There is a dimensional constant $c_0' > 0$ such that if u satisfies (1.3) with c_0', and $0 < r < c_0'$ is sufficiently small that $u(re_n, T) > 0$, then the following property holds:*

$$\int_0^{t_0} \max_{x \in \partial B_a(0)} u(x, t) dt \leq C \int_0^{t_0} u(-ae_n, t) dt \tag{3.2}$$

where $t_0 = t(re_n; u)$, $a = r^\gamma > r$, $\gamma < 1$ and C are dimensional constants. Further suppose u satisfies the first condition of (1.1), and for $0 \leq t \leq t_0$

$$2^{-m_0^2/2} \leq \frac{u(-e_n, t)}{u(-e_n, 0)} \leq 2^{m_0^2/2}$$

where m_0 is the largest integer such that $r \leq 2^{-m_0}$. Then

$$\max_{x \in \partial B_a(0)} u(x, t) \leq C u(-ae_n, t)$$

for $0 \leq t \leq t_0/2$ and a dimensional constant C.

Proof of Proposition 3.1. Let m_0 be the smallest integer such that

$$A \leq 2^{m_0^2/2} \text{ and } 2^{-m_0/\gamma} < c_0'.$$

Then by Lemma 3.2, (3.1) holds for $a = 2^{-m_0}$ $(r = 2^{-m_0/\gamma})$ and small t such that

$$t \leq \min\{t(re_n; u), T\}/2$$

and hence the result follows. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

For the proof of Lemma 3.2, we first show the following two lemmas.

Lemma 3.3. *If $x_0 \in B_2(0)$ and $\min\{u(x, T) : x \in B_s(x_0)\} = 0$, then*

$$\int_0^T u(x_0, t)dt \leq C(n)s^2.$$

Lemma 3.4. *There is a dimensional constant $c_0' > 0$ such that if u satisfies (1.3) with c_0', and $0 < r < c_0'$ is sufficiently small that $u(re_n, T) > 0$, then the following property holds: for $t_0 = t(re_n; u)$,*

$$\Omega_{t_0} \cap B_{2^{-m}}(0) \subset (\Omega_0 + 2^{-5m}e_n) \tag{3.3}$$

where $m \geq 1$ is any integer such that $r < r^\gamma \leq 2^{-m}$, $0 < \gamma < 1$ is a dimensional constant.

Proof of Lemma 3.3: Let

$$r_0 = \inf\{r : \min\{u(x, 0) : x \in B_r(x_0)\} = 0\}.$$

and change the coordinate so that $x_0 = 0$. We construct a radially symmetric barrier as follows: let $\phi(x)$ be the harmonic function in $B_1(0) - B_{1/2}(0)$ with $\phi = 0$ on $\partial B_1(0)$ and $\phi = 1$ on $\partial B_{1/2}(0)$. Let

$$\alpha(t) = C_1 u(0, t) \text{ and } r^2(t) = C(n)\int_0^t \alpha(s)ds + r^2,$$

where $C_1 > 0$ is a dimensional constant to be chosen later. Then

$$h(x, t) := \alpha(t)\phi(r(t)^{-1}x).$$

is a subsolution of (HS), since

$$V = r'(t) \leq |D\phi| = C(n)\alpha(t)r^{-1}(t) \text{ on } \Gamma_t(h) = \partial B_{r(t)}(0).$$

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Now we compare h with u in the domain

$$\Sigma := (I\!\!R^n - B_{r(t)/2}(0)) \times (0, T).$$

At $t = 0$ $h \prec u$ by definition of r and Harnack inequality for u. As long as $\Omega_t(h) \subset \Omega_t(u)$, $h(\cdot, t) \leq u(\cdot, t)$ on $\partial B_{r(t)/2}(0)$ due to the Harnack inequality for u, if C_1 is chosen sufficiently small. Therefore h cannot cross u from below for the first time in $\Omega(h)$. On the other hand, as long as $h(\cdot, t) \leq u(\cdot, t)$ on $\partial B_{r(t)/2}(0)$, $h(\cdot, t) \prec u(\cdot, t)$ due to Theorem 2.7. Thus we conclude that $h \leq u$ in Σ, and in particular

$$B_{r(T)}(0) \subset \Omega_T(h) \subset \Omega_T(u).$$

By our hypothesis $r(T) \leq s$, which means

$$\int_0^T \alpha(t)dt = \int_0^T C_1 u(0, t)dt \leq C(n)s^2.$$

This yields our conclusion. $\qquad\square$

Proof of Lemma 3.4: We prove the lemma by induction for $1 \leq m \leq m_0$, where m_0 is the largest integer such that $r^\gamma \leq 2^{-m_0}$. Suppose (3.3) holds for some $m \in \mathbb{N}$. To simplify notations, denote

$$B_1 = B_{2^{-m}}(0), B_2 = B_{2^{-m} \cdot 4/5}(0), B_3 = B_{2^{-m} \cdot 3/5}(0), B_4 = B_{2^{-m-1}}(0),$$

and $p_0 = -2^{-m-1}e_n$.

Let $h(x, t) = h^t(x)$ be the harmonic function in $(\Omega_0 + 2^{-5m}e_n) \cap B_1$ such that

$$h^t(x) = \begin{cases} 0 & \text{on } \Gamma_0 + 2^{-5m}e_n, \\ \\ u(x, t) & \text{on } \partial B_1. \end{cases}$$

Due to the maximum principle for harmonic functions, $u(\cdot, t) \leq h^t$ in B_1. In particular, for $x \in \Gamma_0 \cap B_2$ and $0 \leq s \leq 2^{-5m}$,

$$\begin{aligned} u(x + se_n, t) &\leq h^t(x + se_n) \\ &\leq C(\frac{2^{-5m}}{2^{-m}})^k h^t(x - \frac{2^{-m}}{10}e_n) \\ &\leq C2^{-4km}2^m \int_{\partial B_1} u(x, t)dx \\ &\leq C2^{-4km}(u(p_0, t) + 2^m \int_S u(x, t)dx) \end{aligned}$$

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where $S = \partial B_1 \cap \{\Gamma_0 + s e_n : 0 \le s \le 2^{-5m}\}$, C and k are positive dimensional constants. Denote the last term of the above inequality by m_t, i.e.,

$$m_t = C 2^{-4km}(u(p_0, t) + 2^m \int_S u(x, t) dx).$$

It then follows from above arguments that

$$u(x + s e_n, t) \le m_t \tag{3.4}$$

for $x \in \Gamma_0 \cap B_2$ and $0 \le s \le 2^{-5m}$. Next, let w_t be the harmonic function in $\Omega_0 \cap B_2$ such that

$$w_t(x) = \begin{cases} u(x, t) & \text{on } \partial B_2 \cap \Omega_0, \\ \\ m_t & \text{on } \Gamma_0 \cap B_2. \end{cases}$$

Then $u(x, t) \le w_t(x)$ in $\Omega_0 \cap B_2$, and by Lemma 2.9

$$\begin{aligned} \max_{x \in \Omega_0 \cap \partial B_3} u(x, t) &\le \max_{x \in \partial B_3} w_t(x) \\ &\le C(m_t + u(p_0, t)). \end{aligned}$$

Above inequality and (3.4) imply

$$\begin{aligned} \max_{x \in \partial B_3} u(x, t) &\le C(m_t + u(p_0, t)) \\ &\le C(2^{-4km+m} \int_S u(x, t) dx + u(p_0, t)). \end{aligned} \tag{3.5}$$

Now for $x_0 \in \Gamma_0 \cap B_4$, construct a wedge Λ_{x_0} with vertex x_0:

$$\Lambda_{x_0} = x_0 + \{(x', x_n) : L|x'| < x_n < \frac{2^{-m}}{10}\},$$

which is contained in $B_3 - \Omega_0$. Let $v(x, t)$ solve (HS) with

$$\begin{cases} v(x, t) = \max_{x \in \partial B_3} u(x, t) & \text{on the fixed boundary } \partial B_3, \\ \\ \Gamma_0(v) = \Lambda_{x_0}, \end{cases}$$

Note that by Theorem 2.7 $u \le v$ for $0 \le t \le t_0$. Hence for

$$\bar{d}(x_0) := \max\{s : u(x_0 + s e_n, t_0) > 0, x_0 + s e_n \in B_4\},$$

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we get

$$
\begin{aligned}
\bar{d}(x_0) \;&\leq\; d(x_0, t_0; v) := \inf\{d : x_0 + de_n \in \Gamma_{t_0}(v)\} \\
&\leq\; C\Big(\int_0^{t_0} \frac{\max_{x\in\partial B_3} u(x,t)}{2^{-m-1}} dt\Big)^\alpha \\
&\leq\; C\Big(2^{-4km+2m}\int_0^{t_0}\int_S u(x,t)dxdt + \int_0^{t_0}\frac{u(p_0,t)}{2^{-m}}dt\Big)^\alpha \quad (3.6)
\end{aligned}
$$

where

$$
S = \partial B_1 \cap \{\Gamma_0 + se_n : 0 \leq s \leq 2^{-5m}\},
$$

the second inequality follows from Lemma 2.13 with a constant $1/2 < \alpha < 1$ depending on n, and the last inequality follows from (3.5).

On the other hand, if (3.3) holds for m, then Lemma 3.3 implies

$$
\int_0^{t_0} u(x,t)dt \leq C2^{-10m} \text{ for } x \in S.
$$

It follows that

$$
\int_0^{t_0}\int_S u(x,t)dxdt \leq C2^{-15m}.
$$

We proceed to find an upper bound of $\displaystyle\int_0^{t_0}\frac{u(p_0,t)}{2^{-m}}dt$: construct a wedge

$$
\tilde{\Lambda}_{x_0} = x_0 + \{(x', x_n) : -\frac{2^{-m}}{10} < x_n < -L|x'|\}
$$

in Ω_0, and a ball $\tilde{B}$ inside $\tilde{\Lambda}_{x_0}$. Let w solve (HS) with

$$
\begin{cases}
w(x,t) = cu(p_0,t) & \text{on the fixed boundary } \partial\tilde{B}, \\[2mm]
\Gamma_0(w) = \tilde{\Lambda}_{x_0}.
\end{cases}
$$

If c is a sufficiently small dimensional constant, then due to Harnack inequality for harmonic functions and Theorem 2.7

$$
w \leq u \text{ for } 0 \leq t \leq t_0.
$$

Due to Lemma 2.13

$$
\Big(\int_0^{t_0}\frac{cu(p_0,t)}{2^{-m}}dt\Big)^\beta \leq d(0,t_0;u) := \inf\{d : de_n \in \Gamma_{t_0}(u)\} = r \qquad (3.7)
$$

for a dimensional constant $\beta > 1$. Let $\gamma = 1/(12\beta)$ and let m_0 be the largest integer such that $r^\gamma \le 2^{-m_0}$. Then if $m < m_0$, (3.6) and (3.7) imply

$$
\begin{aligned}
\bar{d}(x_0) &\le C(2^{-4km-13m} + r^{1/\beta}2^m)^\alpha \\
&\le C(2^{-13m} + 2^{-11m})^{1/2} \\
&\le C2^{-11m/2}.
\end{aligned}
$$

Here C is a dimensional constant, and the second inequality follows from $r^\gamma = r^{1/(12\beta)} < 2^{-m_0} < 2^{-m}$. Thus if $m > m_1 = m_1(n)$, then

$$
\bar{d}(x_0) \le 2^{-5(m+1)}.
$$

For $m \le m_1$, (3.3) holds if we let $c_0' < 2^{-5m_1}$. $\qquad\qquad\square$

We are now ready to prove Lemma 3.2 using Lemma 3.4.

Proof of Lemma 3.2: Let m_0 be the largest integer such that $r^\gamma \le 2^{-m_0}$, where γ is given as in Lemma 3.4. For $0 \le t \le t_0 = t(re_n; u)$ and $1 \le m \le m_0$, denote

$$
A_{t,m} = \max\{u(x,t) : x \in \partial B_{2^{-m}}(0) \cap \{\Gamma_0 + se_n : 0 \le s \le 2^{-5m}\}\}.
$$

Note that, due to (3.5),

$$
\begin{aligned}
A_{t,m+1} &\le \max_{\partial B_{2^{-m} \cdot 3/5}(0)} u(x,t) \\
&\le C\left(2^m \int_{\partial B_{2^{-m}}(0) \cap \{\Gamma_0 + se_n : 0 \le s \le 2^{-5m}\}} u(x,t)dx + u(p_m, t)\right) \\
&\le C(2^{-4m} A_{t,m} + u(p_m, t))
\end{aligned}
$$

where $p_m = -2^{-m-1}e_n$ and C is a dimensional constant. Then by iteration on m we obtain that

$$
A_{t,m_0} \le C^{m_0} 2^{-2m_0^2} A_{t,1} + \sum_{m=1}^{m_0} C^{m_0-m} 2^{-2(m_0^2-m^2)} u(p_m, t).
$$

Here, by Harnack inequality, $u(p_m, t)$ is bounded as follows:

$$
u(p_m, t) \le C_0^{m_0-m} u(p_{m_0}, t) \tag{3.8}
$$

where C_0 is a constant depending on n. Hence, if $m_0 \ge \bar{m}$ for a sufficiently large constant $\bar{m}$ depending on n and C_0, then

$$
A_{t,m_0} \le 2^{-m_0^2}(A_{t,1} + u(p_{m_0}, t)) \tag{3.9}
$$

where $p_{m_0} = -2^{-m_0-1}e_n$. Observe that

$$\int_0^{t_0} A_{t,1} dt \leq \int_0^{t_0} \max_{x \in B_{1/2}(0)} u(x,t) dt \leq C \int_{\partial B_1(0)} \int_0^{t_0} u(x,t) dt dx \leq C \quad (3.10)$$

where the last inequality follows from Lemma 3.3.

Next let

$$a = 2^{-m_0-1} \approx r^\gamma. \quad (3.11)$$

Then by (3.9) and (3.10)

$$\int_0^{t_0} \max_{x \in \partial B_a(0)} u(x,t) dt \leq 2^{-m_0^2} \int_0^{t_0} A_{t,1} dt + C \int_0^{t_0} u(-ae_n, t) dt$$

$$\leq C(2^{-m_0^2} + \int_0^{t_0} u(-ae_n, t) dt). \quad (3.12)$$

It follows that

$$2^{(-m_0-1)/\gamma} \leq r$$

$$\leq C(\int_0^{t_0} \frac{\max\limits_{x \in \partial B_a(0)} u(x,t)}{a} dt)^\alpha$$

$$\leq C(2^{-m_0^2+m_0} + 2^{m_0} \int_0^{t_0} u(-ae_n, t) dt)^\alpha$$

$$\leq C2^{(-m_0^2+m_0)\alpha} + C(2^{m_0} \int_0^{t_0} u(-ae_n, t) dt)^\alpha \quad (3.13)$$

where the second inequality follows from Lemma 2.13 with $1/2 < \alpha < 1$ depending on n, and the third inequality follows from (3.11) and (3.12). If $m_0 \geq \bar{m}(n) >> 1$,

$$C2^{(-m_0^2+m_0)\alpha+1} \leq 2^{(-m_0-1)/\gamma}$$

where $C > 1$ is a constant given in (3.13). Thus (3.13) implies

$$2^{(-m_0-1)/\gamma} \leq C(2^{m_0} \int_0^{t_0} u(-ae_n, t) dt)^\alpha$$

and we get

$$2^{(-m_0^2+m_0)\alpha} \leq 2^{(-m_0-1)/\gamma} \leq C(2^{m_0} \int_0^{t_0} u(-ae_n, t) dt)^\alpha. \quad (3.14)$$

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By (3.12) and (3.14)

$$\int_0^{t_0} \max_{x \in \partial B_a(0)} u(x,t)dt \leq C(2^{-m_0^2} + \int_0^{t_0} u(-ae_n,t)dt)$$

$$\leq C\int_0^{t_0} u(-ae_n,t)dt.$$

To prove the second part of the lemma, we further assume $u(x,s) \leq 10u(x,t)$ for $0 \leq s < t \leq T$, and

$$1/10 \leq \frac{u(-e_n,t)}{u(-e_n,0)} \leq \tilde{M} := 2^{m_0^2/2}$$

for $0 \leq t \leq t_0 = t(re_n;u)$. Then by Harnack inequality for harmonic functions

$$\max_{0 \leq t \leq t_0} u(-ae_n,t) \leq 10u(-ae_n,t_0) \leq C^{m_0}\tilde{M}$$

where $a = 2^{-m_0-1}$ and C is a dimensional constant. Hence by Lemma 2.13

$$(\int_0^{t_0} \frac{C^{m_0}\tilde{M}}{a}dt)^\beta \geq r$$

for a constant $0 < \beta < 1$ depending on n, which implies

$$\frac{1}{t_0} \leq C^{m_0}2^{m_0}r^{-1/\beta}\tilde{M} \leq 2^{km_0}\tilde{M}$$

where k is a positive dimensional constant.

Since

$$10A_{s,1} \geq A_{t_0/2,1} \text{ for } s \geq t_0/2, \text{ and } \int_0^{t_0} A_{t,1}dt \leq C,$$

it follows that

$$\frac{t_0}{20}A_{t_0/2,1} \leq \int_{t_0/2}^{t_0} A_{t,1}dt \leq C.$$

Hence if $t \leq t_0/2$, then

$$A_{t,1} \leq 10A_{t_0/2,1} \leq C\frac{1}{t_0} \leq C2^{km_0}\tilde{M}$$

for positive dimensional constants C and k. Then by (3.9)

$$A_{t,m_0} \leq C2^{-m_0^2}(2^{km_0}\tilde{M} + u(p_{m_0},t)) \tag{3.15}$$

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and, due to Harnack inequality,

$$u(p_{m_0}, t) \geq C2^{-\tilde{k}m_0} u(-e_n/2, t) \geq C2^{-\tilde{k}m_0}$$

for $\tilde{k} = \tilde{k}(n) > 0$. Since $\tilde{M} = 2^{m_0^2/2}$, if $m_0 \geq \bar{m}$ for a sufficiently large $\bar{m}$ depending on n, then (3.15) implies

$$A_{t,m_0} \leq Cu(p_{m_0}, t)$$

and hence for $a = 2^{-m_0-1}$

$$\max_{x \in \partial B_a(0)} u(x, t) \leq Cu(-ae_n, t).$$

$\square$

4 ϵ-flatness of level sets of u in a ball

In this section, we show the flatness property of u satisfying the hypothesis of Proposition 3.1, i.e., satisfying the first condition of (1.1), (1.2) and (1.3). $0 < r < c_0$ will denote a sufficiently small constant such that $u(re_n, T/2) > 0$.

To show the ϵ-flatness of u (Proposition 4.1), we will construct a star-shaped barrier w for u. Proof of the proposition will need more arguments than that of Lemma 3.2 in [CJK] due to the fact that the Hölder continuity of u in time (Corollary 2.3 in [CJK]) does not hold here.

We begin by introducing some notations.

$\diamond$ For nonzero vectors v_1, v_2 in $I\!\!R^n$, we define $\alpha(v_1, v_2)$ to be the (smaller) angle between v_1 and v_2.

$\diamond$ We say that a function f has a *cone of monotonicity*

$$W(\theta, \nu) := \{p \in I\!\!R^n : \alpha(p, \nu) < \theta\}$$

or f is *monotone for the cone* $W(\theta, \nu)$ in $D \subset I\!\!R^n$ if f is monotone increasing along every direction $p \in W(\theta, \nu)$ in D.

Consider a star-shaped domain Ω' such that

(i) $\Omega' \cap B_2(0) = \Omega_0 \cap B_2(0)$;

(ii) Ω' is star-shaped with respect to every $x \in K' \subset \Omega'$ for a sufficiently large ball K'.

Let v_0 be the harmonic function in $\Omega' - K'$ with data 1 on $\partial K'$ and 0 on $\partial \Omega'$, and let v be the solution of (HS) with $v(x, 0) = v_0(x)$ and $v(x, t) = 1$ on $\partial K'$. It follows from comparison arguments (see Lemma 3.3 in [CJK]) that v is starshaped with respect to K'. In particular, $v(\cdot, t)$ is monotone in $B_1(0)$ for the cone $W(\theta - \epsilon, -e_n)$ for $0 \le t \le t(e_n; v)$, if K' is sufficiently large depending on ϵ. Here ϵ is a small constant to be chosen later.

Based on v, we will construct a supersolution w^{sup} and a subsolution w^{sub} of (HS) such that in $B_r(0)$, $w^{sub} \le u \le w^{sup}$ and the level sets of w^{sub} and w^{sup} are close to each other.

We begin by constructing a concentric ball B_2 of $B_1(0)$ as follows: Let k_0, k_1 and k_2 be sufficiently large numbers satisfying

$$k_0 \gg k_1, \quad \gamma = \frac{k_0}{k_0 + k_1 + k_2}$$

where γ is a constant given as in Lemma 3.4. (k_0, k_1 and k_2 will be determined later in the proof.) Let $k = k_0 + k_1 + k_2$ and let $\epsilon = r^{1/k}$ ($\epsilon^k = r$). We denote

$$B_2 := B_{\epsilon^{k_0}}(0) \subset B_1(0) =: B_1.$$

Since $r^\gamma = \epsilon^{k_0}$, Lemma 3.4 implies

$$\Omega_{t_0} \cap B_2 \subset (\Omega_0 + \epsilon^{5k_0} e_n) \cap B_2$$

for $t_0 := t(re_n; u)$. Next define

$$H = (\Gamma_0 - \epsilon^{k_0 + k_1} e_n) \cap \frac{2}{3} B_2$$

where $\frac{2}{3} B_2 = B_{2\epsilon^{k_0}/3}(0)$. Observe that

$$\text{radius}(B_2) \gg \text{dist}(H, \Gamma_0) \gg \max_{x \in \Gamma_{t_0} \cap B_2} \text{dist}(x, \Gamma_0).$$

If k_1 is sufficiently large, Lemma 2.10 implies that for any positive harmonic functions h and g in $B_2 \cap \Omega_0$, vanishing on Γ_0, there exists a constant $c > 0$ such that

$$1 - \epsilon \le \frac{h}{cg} \le 1 + \epsilon \text{ on } H. \tag{4.1}$$

Let $h(x, t) := h^t(x)$ be the harmonic function in $B_2 \cap \Omega_0$ such that

$$h^t = \begin{cases} 0 & \text{on } \Gamma_0, \\ \\ u(x, t) & \text{on } \partial B_2 \cap \Omega_0. \end{cases} \tag{4.2}$$

Then
$$h^t(x) \leq u(x,t) \text{ in } B_2.$$

Since $h := h^t$ and $g := v_0$ are both positive harmonic functions in $B_2 \cap \Omega_0$, which vanish on Γ_0, h and g satisfies (4.1) with $c = c_t$, i.e., there is a constant c_t such that

$$1 - \epsilon \leq \frac{h^t}{c_t v_0} \leq 1 + \epsilon \text{ on } H. \tag{4.3}$$

Hence we get

$$(1 - \epsilon)c_t v_0 \leq u(x,t) \text{ for } x \in H. \tag{4.4}$$

Similarly,let $\tilde{h}(x,t) := \tilde{h}^t(x)$ be a harmonic function in $B_2 \cap (\Omega_0 + \epsilon^{5k_0}e_n)$ such that

$$\tilde{h}^t = \begin{cases} 0 & \text{on } \Gamma_0 + \epsilon^{5k_0}e_n, \\[2mm] u(x,t) & \text{on } \partial B_2 \cap (\Omega_0 + \epsilon^{5k_0}e_n). \end{cases} \tag{4.5}$$

Then

$$u(x,t) \leq \tilde{h}^t(x) \text{ in } B_2. \tag{4.6}$$

Since $\epsilon^{5k_0} \ll \epsilon^{k_0+k_1} = \text{dist}(H, \Gamma_0)$, from a similar argument as for h^t, we obtain a constant $\tilde{c}_t$ such that

$$1 - \epsilon \leq \frac{\tilde{h}^t}{\tilde{c}_t v_0} \leq 1 + \epsilon \text{ on } H. \tag{4.7}$$

Therefore we have

$$u(x,t) \leq \tilde{h}^t(x) \leq (1 + \epsilon)\tilde{c}_t v_0(x) \text{ for } x \in H. \tag{4.8}$$

Let w be the star-shaped solution of (HS) with

$$\begin{cases} \Omega_0(w) = \Omega_0(v), \\[2mm] w(x,t) = (1 - 3\epsilon)c_t \text{ on } \partial K'. \end{cases}$$

In other words,

$$w(x,t) = (1 - 3\epsilon)c_t v(x, (1 - 3\epsilon) \int_0^t c_s ds).$$

We now state our main proposition:

Proposition 4.1. *Suppose u satisfies the first condition of (1.1), (1.2) and (1.3) with $c_0 > 0$ depending only on dimension and on $A > 0$. Suppose $0 < r < c_0$ is sufficiently small that $u(re_n, T/2) > 0$, then the following property holds: there is an integer $k > 0$ depending on dimension such that if $0 < s \le r$ and ϵ satisfies $\epsilon^k = s$, then the level sets of $u(\cdot, t)$ stay $s\epsilon$-close to those of w in $B_{2s}(0)$ for $0 \le t \le t(se_n; u)$. In particular u is $s\epsilon$-monotone for the cone $W(\theta - \epsilon, -e_n)$ in $B_{2s}(0)$ for $0 \le t \le t(se_n; u)$.*

Proof. Without loss of generality, we may assume $s = r$. We will perturb w by order of ϵ to construct w^{sup} and w^{sub}.

Let $\tilde{w}$ be the star-shaped solution of (HS) with

$$\begin{cases} \Omega_0(\tilde{w}) = \Omega_0(v), \\[2mm] \tilde{w}(x, t) = (1 + 3\epsilon)\tilde{c}_t \text{ on } \partial K', \end{cases}$$

namely

$$\tilde{w}(x, t) = (1 + 3\epsilon)\tilde{c}_t v(x, (1 + 3\epsilon) \int_0^t \tilde{c}_s ds).$$

Since $r = \epsilon^k \ll \epsilon^{k_0 + k_1}$, Corollary 2.3 in [CJK] implies

$$v(x, t_0') \le (1 + \epsilon)v_0(x) \text{ for } x \in H$$

where $t_0' := t(re_n; v) = t(\epsilon^k e_n; v)$. Thus for $0 \le s \le r$, $x \in H$ and $t = t(se_n; w)$,

$$\begin{aligned} w(x, t) &= (1 - 3\epsilon)c_t v(x, t(se_n; v)) \\ &\le (1 - 2\epsilon)c_t v_0(x) \\ &\le (1 - \epsilon)u(x, t), \end{aligned} \tag{4.9}$$

where the last inequality follows from (4.4). Similarly if $x \in H$ and $t = t(se_n; \tilde{w})$ then

$$\begin{aligned} \tilde{w}(x, t) &= (1 + 3\epsilon)\tilde{c}_t v(x, t(se_n; v)) \\ &\ge (1 + 3\epsilon)\tilde{c}_t v_0(x) \\ &\ge (1 + \epsilon)u(x, t) \end{aligned} \tag{4.10}$$

where the last inequality follows from (4.8).

Based on $\tilde{w}$ we construct a supersolution w^{sup} as in Lemma 3.2 of [CJK], by bending the free boundaries of $\tilde{w}$ up above $\Gamma_t(u)$ in $\frac{2}{3}B_2 - \frac{1}{3}B_2$. Define

$$\psi(x) = \frac{1}{|x|^2} \cdot (x_1, ..., x_{n-1}, -x_n + 2|x|^2)$$

where $x = (x_1, ..., x_n) \in I\!\!R^n$. Since ψ is the composition of reflection about $x_n = 1$ and the conformal mapping sending x to $\dfrac{x}{|x|^2}$,

$$\phi := \psi(x + e_n) - e_n$$

is also conformal. Observe that ϕ fixes 0 and will bend the free boundary of $\tilde{w}$ up above $\Gamma_t(u)$ in $\frac{2}{3}B_2 - \frac{1}{3}B_2$. Set S_t to be the strip between H and $\Gamma_t(\tilde{w})$ and define

$$w^{sup}(x,t) = (1 + \epsilon)\tilde{w}(\phi^{-1}(x),t) \text{ in } R := \phi(S_t) \cap \frac{2}{3}B_2.$$

Note that $w^{\mathrm{sup}}(\cdot,t)$ is harmonic in R and

$$\Gamma_t(w^{sup}) = \phi(\Gamma_t(\tilde{w}))$$

in $\frac{2}{3}B_2$. By a similar argument as in Lemma 3.2 of [CJK], w^{sup} is a super-solution in $\frac{2}{3}B_2 \times [0,T]$.

Now we compare w^{sup} and u in

$$\Sigma := (\frac{1}{2}B_2 \cap R) \times [0, \tilde{T}],$$

where $\tilde{T} = \min\{t_0, t(re_n; w^{sup})\}$, $t_0 = t(re_n; u)$.

First note that, due to (4.6), (4.10) and the fact

$$\max_{\phi^{-1}(R)} |\phi(x) - x| \approx \epsilon^{2k_0} \ll \epsilon^{k_0+k_1} \approx \mathrm{dist}(H, \Gamma_0(u)), \qquad (4.11)$$

we have

$$u(x,t) \le w^{sup}(x,t) \text{ on } \phi(H) \cap \frac{1}{2}B_2 \times [0,\tilde{T}].$$

Secondly observe that in $\frac{2}{3}B_2 - \frac{1}{3}B_2$ and for $0 \le t \le \tilde{T}$

$$\Gamma(u) \subset (\bigcup_{s \le \epsilon^{5k_0}} (\Gamma_0(u) + se_n)) \times [0,\tilde{T}]$$

due to Lemma 3.4. On the other hand,

$$\Gamma_t(w^{sup}) = \phi(\Gamma_t(\tilde{w})) \subset \bigcup_{s \ge \epsilon^{2k_0}} (\Gamma_0(\tilde{w}) + se_n) = \bigcup_{s \ge \epsilon^{2k_0}} (\Gamma_0(u) + se_n).$$

Hence it follows that

$$\Omega(u) + \epsilon^{2k_0}/2e_n \subset \Omega(w^{sup}) \text{ in } (\frac{2}{3}B_2 - \frac{1}{3}B_2) \times [0,\tilde{T}] \qquad (4.12)$$

Due to (4.10), (4.12) and Lemma 2.10, we obtain

$$u(x,t) \le w^{sup}(x,t) \text{ on } \partial(\frac{1}{2}B_2 \cap R) \times [0,\tilde{T}].$$

Therefore $u \le w^{sup}$ on the parabolic boundary of Σ, and Theorem 2.7 yields that

$$u \le w^{sup} \text{ in } \Sigma. \tag{4.13}$$

Similarly we construct a subsolution w^{sub} as follows. Let $\bar{\psi}$ be the composition of reflection about $x_n = 1$ and the conformal mapping sending x to $\dfrac{x}{|x-2e_n|^2}$, then $\bar{\phi} := \bar{\psi}(x+e_n) - e_n$ will bend the free boundary of w down below $\Gamma_t(u)$ in $\frac{2}{3}B_2 - \frac{1}{3}B_2$. Let us define

$$w^{sub}(x,t) = (1-\epsilon)w(\bar{\phi}^{-1}(x),t)$$

Then $w^{sub}(\cdot,t)$ is harmonic in $\bar{R} := \frac{2}{3}B_2 \cap \bar{\phi}(\bar{S}_t)$, where $\bar{S}_t$ is defined similarly as in S_t. From an argument parallel to the one for showing (4.13), we get

$$w^{sub} \le u \in (\frac{1}{2}B_2 \cap \bar{R}) \times [0,T'],$$

where $T' = \min\{t_0, t(re_n; w^{sub})\}$.

Let $B_3 = B_{2r}(0) = B_{2\epsilon^k}(0)$. Then in B_3, if $0 \le t \le T'$ and $0 < s \le r$, the level sets $\{x \in B_3 : u(x,t) = s\}$ are located between $\{x \in B_3 : w^{sup}(x,t) = s\}$ and $\{x \in B_3 : w^{sub}(x,t) = s\}$, which are in the ϵr-neighborhood of $\{x \in B_3 : \tilde{w}(x,t) = s\}$ and $\{x \in B_3 : w(x,t) = s\}$, respectively. Since w is monotone in $B_1(0)$ for the cone $W(\theta - \epsilon, -e_n)$, to conclude it suffices to prove that the level sets $\{x \in B_3 : w(x,t) = s\}$ are contained in the $r\epsilon$-neighborhood of the level sets $\{x \in B_3 : \tilde{w}(x,t) = s\}$ of $\tilde{w}$.

Recall that w and $\tilde{w}$ are star-shaped solutions with the same initial domain, and with different fixed boundary values $(1-\epsilon)c_t$ and $(1+\epsilon)\tilde{c}_t$, respectively. By (4.4) and (4.8), c_t and $\tilde{c}_t$ satisfy

$$1 \le \frac{\tilde{c}_t}{c_t} \le (1+2\epsilon)\frac{\tilde{h}^t(x)}{h^t(x)} \tag{4.14}$$

for $x \in H$. Let $x_0 = -\epsilon^{k_0+k_1}e_n \in H$, then

$$\tilde{h}^t(x_0) \le (1 + C\frac{\max_{x \in \partial B_2} u(x,t)}{u(-\epsilon^{k_0}e_n, t)}\epsilon^{\beta k_0})h^t(x_0)$$

for $C > 0$ and $\beta > 0$ depending on n, since $h_t = \tilde{h}_t = u(x, t)$ on $\partial B_2 \cap \{h_t > 0\}$, and since

$$\Gamma_t(\tilde{h}) = \Gamma_t(h) + \epsilon^{5k_0} e_n \text{ in } B_2,$$

with ϵ^{5k_0} much smaller than $\epsilon^{k_0+k_1} = d(x_0, \Gamma_t(h))$ for $0 \leq t \leq T'$. Therefore if k_0 is sufficiently large, (4.14) implies

$$1 \leq \frac{\tilde{c}_t}{c_t} \leq 1 + \frac{\max_{x \in \partial B_2} u(x, t)}{u(-\epsilon^{k_0} e_n, t)} \epsilon. \tag{4.15}$$

By Proposition 3.1,

$$\max_{x \in \partial B_2} u(x, t) \leq C u(-\epsilon^{k_0} e_n, t)$$

and thus

$$1 \leq \frac{\tilde{c}_t}{c_t} \leq 1 + C\epsilon$$

for a dimensional constant C. Now (4.16) and Lemma 3.3 of [CJK] yields that the level sets of w and $\tilde{w}$ are contained in the $Cr\epsilon$-neighborhood of each other.

$\square$

5 Non-degeneracy and Lipschitz continuity

Our next goal is to improve Proposition 4.1 to conclude that $u(\cdot, t)$ is Lipschitz in space in $B_r(0)$ for $0 \leq t \leq t(re_n; u)$. For this purpose we first need to show that u is non-degenerate on $\Gamma(u)$ in ϵ-scale (Proposition 5.1).

Proposition 5.1. *Let u be a solution of (HS) with $0 \in \Gamma$ in $B_2(0) \times \mathbb{R}^+$. In addition suppose that $t(e_n; u) = 1$, u satisfies (1.1) with $A \leq 1$ and $u(\cdot, t)$ is ϵ-monotone for $0 < \epsilon < \epsilon_n$ for the cone $W(\theta, -e_n)$, $\theta > \pi/4$ in the region $B_2(e_n)$ for $t \in [0, 1]$, where ϵ_n is a dimensional constant and $t_0 = t(e_n)$. Then there exists a dimensional $C > 0$ such that*

$$B_{1/2}(e_n) \cap (\Omega_t + se_n) \subset \Omega_{(1+C\epsilon)t} \tag{5.1}$$

for $t(e_n/2; u) \leq t \leq t(e_n; u)$ and $0 \leq s \leq \epsilon$.

The proof is parallel to that of Proposition 4.1 in [CJK], but we state the proof below to illustrate the use of (1.1).

Proof. For $P_1 \in B_{1/2}(e_n) \cap \Omega_t(u)$, denote $t_1 = t(P_1)$. The goal is to show that $u(P_1 + se_n, (1 + C\epsilon)t_1) > 0$ for some $C(n) > 0$. Let $P_2 = P_1 + \frac{1}{5}e_n, R = B_{1/2}(P_2) - B_{1/10}(P_2)$ and $\Sigma = R \times [0, t_1]$. Define the function

$$w(x, t) := \inf_{B_{\epsilon\varphi(x)}(x)} u(y - \epsilon e_n, t)$$

where φ defined in R satisfies the following properties:

$(a) \quad \Delta(\varphi^{-Q_n}) = 0 \quad$ in R;

$(b) \quad \varphi = A_n \qquad$ on $\partial B_{1/10}(P_2)$;

$(c) \quad \varphi = 1/\sqrt{2} \qquad$ in $\partial B_{1/2}(P_2)$.

Here $Q_n > 0$ is a dimensional constant chosen such that Lemma 9 in [C1] applies, A_n is chosen sufficiently large that $\varphi(P_1) > 2$. Note that also $|D\varphi| \leq C$ where C depends on Q_n and A_n.

Now we compare w and u in Σ. First due to the ϵ-monotonicity of u (in space) we can argue as in [CJK] to check that $w \geq u$ on the parabolic boundary of Σ. Observe that, formally speaking, w satisfies

$$\frac{w_t}{|Dw|} \geq (1 - O(\epsilon))|Dw| \text{ on } \Gamma(w) \cap \Sigma,$$

due to the definition of w. It follows that for some constant $C = C(n) > 0$

$$w_1(x, t) := (1 + C\epsilon)w(x, (1 + C\epsilon)t) \text{ is a supersolution of (HS) in } \Sigma,$$

(For rigorous proof see the proof of Proposition 4.1 in [CJK].)

Note that $t_1 \leq 1$. Thus if u satisfies (1.1) with $A < 1$ then $w \leq w_1(x, t)$ and therefore Theorem 2.7 yields

$$u(x, t) \leq w_1(x, t) \text{ in } \Sigma.$$

On the other hand by (c) and the definition of w, $w(x, t) \leq u(x + se_n, t)$ at P for $0 \leq s \leq \epsilon$ and thus we can conclude.

$\square$

The following corollary and its proof correspond to Corollary 4.4 in [CJK].

Corollary 5.2. *Let u, ϵ be as given in Proposition 5.1 and let $t = t(e_n; u)$. Then there exists $C(n) > 0$ such that for $x \in \Gamma_t(u) \cap B_{1/2}(e_n)$,*

$$\sup_{y \in B_{2\epsilon}(x)} u_1(y, t) \geq C\epsilon.$$

In terms of our original u, the following holds:

Corollary 5.3. *Let $P_0 \in \Gamma_{t_0}(u)$ and $d(P_0, P_1) = r$, $t_1 = t(P_1; u) > t_0$. Furthermore suppose that u is $r\epsilon$-monotone for $0 < \epsilon < \epsilon_n$ for the cone $W(\theta, e_n), \theta > \pi/4$ in $B_{2r}(P_0) \times [t_0, t_1]$. Then there exits a dimensional constant $C > 0$ such that*

$$\sup_{y \in B_{2r\epsilon}(x)} u(y, t) \geq \frac{C}{t_1 - t_0} r\epsilon.$$

for $t = t_1$.

Remark. Corollary 5.3 states that the lower bound on the normal velocity, or $|Du|$ on $\Gamma(u)$, is proportional to the distance d it has moved in e_n direction from its initial position. Furthermore this lower bound is obtained uniformly in the space neighborhood of size d, suggesting that the regularization of the free boundary occurs in scale of d in space and $t(de_n; u)$ in time.

Proof. Let
$$\tilde{u}(x) = \frac{(t_1 - t_0)}{r} u(rx, (t_1 - t_0)t + t_0)$$
and apply Corollary 5.2 to $\tilde{u}$. $\qquad\square$

The following lemma is a modification of Theorem 5.1 in [CJK]. The proof is parallel to that in [CJK].

Lemma 5.4. *Suppose u solves (HS) in $B_3(0) \times [0, t_0], t_0 = t(e_n; u)$ such that*

(a) *$u(\cdot, t)$ is ϵ-monotone for a cone $W(\theta, -e_n)$ in $B_2(e_n)$ for $0 \leq t \leq t_0$.*

(b) *u satisfies*
$$\sup_{y \in B_{2\epsilon}(x)} u(y, t) \geq C\epsilon \qquad (5.2)$$

for $x \in \Gamma_t(u)$ for $0 \leq t \leq t_0$.

If $\pi/2-\theta < \tan^{-1}(L_n)$ for a dimensional constant $L_n > 0$ and $0 < \epsilon < 1/100$ then there is a constant $0 < \lambda < 1$ depending only on C and n such that $u(\cdot, t)$ is $\lambda\epsilon$-monotone for $W(\theta', -e_n)$ in $B_{2-\epsilon^{1/4}}(e_n)$ for $\epsilon^{1/6} \le t \le t_0$ with $\theta' = \theta - \epsilon^{1/14}$.

Corollary 5.5. *Let u be as in Proposition 5.1. Then $\Gamma_t(u)$ is Lipschitz in space with Lipschitz constant $L < L' = L + C\epsilon^{1/14} < L_n$ in $B_2(0) \times [1/2, 1]$, where C is a dimensional constant.*

Proof. 1. Due to Corollary 5.3, (5.2) holds for u in $B_2(0) \times [1/4, 1]$ with dimensional constant C. By iterating Lemma 5.4 and Proposition 5.1, it follows that $u(\cdot, t)$ is monotone in the cone $W(\theta', -e_n)$ in $B_s(e_n)$ for $b \le t \le 1$, where

$$s = 2 - C\epsilon^{1/4}\Sigma_{k=0}^{\infty}\lambda^{k/4}, \quad b = c + C\epsilon^{1/6}\Sigma_{k=0}^{\infty}\lambda^{k/6}$$

for $C = C(n, A)$, a dimensional constant $0 < c < 1$, and

$$\theta' = \theta - \epsilon^{1/14}\Sigma_{k=0}^{\infty}\lambda^{k/14}. \tag{5.3}$$

Hence if $a(r, A)$ is chosen sufficiently small such that the corresponding ϵ in Proposition 4.1 is sufficiently small, the lemma holds for $1/2 \le t \le 1$. $\qquad\square$

Corollary 5.6. *Let u, r and k as given in Proposition 4.1. Then for $0 < s < r$, u is Lipschitz in space with Lipschitz constant $L < L' + s^{(14k)^{-1}} < L_n$ in $B_s(0) \times [0, t(se_n; u)]$.*

Proof. For any $0 < a \le r$ and $x_0 \in \Gamma_0 \cap B_2(0)$, let

$$\tilde{u}(x, t) = \frac{t(a)}{a}u(a(x - x_0), t(r)t), t(r) = t(x_0 + ae_n; u).$$

We then apply Proposition 4.1 and Corollary 5.5 with $\tilde{u}$ to derive the lemma for $\frac{1}{2}t_0 \le t \le t_0$. Since r is arbitrary small, we get the Lipschitz property of $\Gamma_t(u)$ in $B_a(0)$ for $0 < t < t(ae_n; u)$. $\qquad\square$

Corollary 5.7. *Let u and r be as given in Proposition 4.1. Then for $0 \le t \le t(se_n; u)$ with $0 \le s \le r$,*

$$1/C_1 \le \frac{u(-se_n, t)}{u(-se_n, 0)} \le C_1$$

and

$$\frac{s^2}{C_1u(-se_n, 0)} \le t(se_n; u) \le C_1\frac{s^2}{u(-se_n, 0)},$$

where $C_1 = C_1(n, A)$.

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Proof. Due to Corollary 5.6 u is Lipschitz in space in $B_r(0) \times [0, t(re_n; u)]$ with $r = r(n, A)$. Hence it follows that $\Gamma_t(u)$ is Lipschitz in $B_r(0)$ for $0 < t < t(re_n; u)$, and which implies

$$\max_{\partial B_r(0)} u(x, t) \le Cu(-re_n, t)$$

where $0 < t < T$ and C is a dimensional constant. Furthermore, since $u(-e_n, t) \le Au(-e_n, 0)$, By Harnack inequality

$$\max_{\partial B_s(0)} u(x, t) \le C_1 u(-se_n, 0)$$

for $0 < s < r$, $0 < t < t(se_n; u)$ and $C_1 = C_1(n, A)$. Then the results follow from a similar argument as in Theorem 2.1 and Corollary 2.2 of [CJK]. $\square$

6   Regularity in space

In section 6-7 we modify iteration argument used in section 9-10 of [CJK] to yield further regularity of $\Gamma(u)$.

The main idea is that the nice properties of u in the positive phase propagates to the free boundary over time. In particular in [CJK] $u_t \le C|Du|^2$ in the positive phase, due to the fact that $\Omega_t(u)$ was a globally Lipschitz domain with Lipschitz constant $L < L_n$. This property, ensures that the direction of the spatial gradient $Du(-e_n, t)$, which represents the direction of propagation of the flow in unit scale, does not change too quickly in time (see Lemma 9.1 in [CJK]).

In our case u_t is no longer bounded since $\Omega_t(u)$ may not be Lipschitz outside of $B_1(0)$. Hence we need a new argument to control the change of $\frac{Du}{|Du|}$ over time.

Lemma 6.1. *Let u solve (HS) in $B_1(0) \times (-1, 1)$ with (1.1), $0 \in \Gamma_0(u)$ and $t(e_n; u) = 1$. Suppose*

(a) *$u(\cdot, t)$ is monotone for the cone $W(e_n, \theta_0)$ in $B_1(0)$, with $\theta_0 > \pi/2 - tan^{-1}(L_n)$.*

(b) *$u(\cdot, t)$ is monotone for the cone $W(\nu_l, \theta_l)$ in $B_{2^{-l+1}}(0) \times (0, 2^{-l}\delta_l)$ with $\delta_l = \pi/2 - \theta_l > l^{-2}$.*

Then for integer $l > l_0$ where l_0 depending only on θ_0 and n, there exist a unit vector $\nu_{l+1} \in \mathbb{R}^n$ and $0 < h_0(n) < 1$ and $0 < r_0(n) < 1$ such that

$u(\cdot, t)$ *is monotone increasing in* $B_{2^{-l-5}}(-2^{-l-1}e_n) \times (0, r_0 2^{-l}\delta_l)$ *for the cone* $\eta \in W(\nu_{l+1}, \theta_{l+1})$ *with*

$$\delta_{l+1} \le h_0 \delta_l.$$

Proof. 1. Let u and w be given as in Proposition 4.1. Then the following is true due to Proposition 4.1:

$$|u - w|(x, t) \le C 2^{-\alpha l} w(-2^{-l}e_n, t) \text{ in } B_{2^{-l+1}}(0) \times [0, t(2^{-l}e_n; u)] \quad (6.1)$$

where $C > 0$ and $0 < \alpha < 1$ only depend on n.

Due to the hypothesis and (6.1), $w_2(\cdot, t)$ is $2^{(-\alpha-1)l}$-monotone for the cone $W(e_n, \theta_l)$ in $B_{2^{-l+1}}(0) \times [0, t(2^{-l}e_n; w)]$, and thus due to Corollary 5.2 in [CJK], $w(\cdot, t)$ is monotone for the cone $W(\nu_{l+1}, \theta'_l)$ with

$$\theta'_l = \theta_l - C 2^{-\alpha/14 l}$$

where C only depends on θ_0, n. Let $\delta'_l = \pi/2 - \theta'_l$. Observe that, due to (1.1),

$$t(re_n; u) \ge r t(e_n; u) = r/2.$$

In particular

$$t(2^{-l}e_n; w) \ge (1 - 2^{-\alpha l}t(2^{-l}e_n; u) \ge 2^{-l-1}.$$

Note that due to Lemma 9.1 in [CJK] applied to w, there exist a unit vector $\nu_l \in \mathbb{R}^n$, $0 < r_1(n) < 1$ and $0 < h_0 = h_0(n) < 1$ such that

$$\alpha(Dw_2, \nu_l) \le h_0 \delta'_l \text{ in } B_{2^{-l-3}}(-2^{-l}e_n) \times [0, r_1 2^{-l}\delta_l]. \quad (6.2)$$

Since $\delta_l > l^{-2}$, $h_0 \delta'_l < \tilde{h}_0 \delta_l$ for sufficiently large l, where

$$0 < \tilde{h}_0 = 1/2 + h_0/2 < 1.$$

2. Take a unit vector $p \in \mathbb{R}^n$ such that

$$\alpha(p, \nu_l) \le \pi/2 - (2 - \tilde{h}_0)\delta_l.$$

Due to (6.2), for $0 < \epsilon < 1$ and $0 \le t \le r_0 2^{-l}\delta_l$, there exists $c_0 > 0$ depending on n such that

$$w(\cdot + 2^{-l}\epsilon p, t) \ge c_0 2^{-l}\epsilon(1 - \tilde{h}_0)\delta_l \frac{w(-2^l e_n, t)}{2^{-l}} + w(\cdot, t) \quad (6.3)$$

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in $B_{2^{-l-3}}(-2^{-l}e_n)$ for $0 \leq t \leq r_1 2^{-l}\delta_l$. Now by (6.1),

$$
\begin{aligned}
u(\cdot + 2^{-l}\epsilon p, t) &\geq (1 - C2^{-\alpha l})w(\cdot + 2^{-l}\epsilon p, t) \\
&\geq (c_0(1 - \tilde{h}_0)\epsilon\delta_l - C2^{-\alpha l})w(-2^l e_n, t) + w(\cdot, t) \\
&\geq (1 - 2^{-\alpha l})(c_0\tilde{h}_0\epsilon\delta_l - 2C2^{-\alpha l})u(-2^{-l}e_n, t) + u(\cdot, t),
\end{aligned}
$$

in $B_{2^{-l-3}}(-2^{-l}e_n)$ for $0 \leq t \leq r_1 2^{-l}\delta_l$, where the second inequality is due to (6.3). In other words $u(\cdot, t)$ is $2^{-l}\epsilon$-monotone in the direction of p in $B_{2^{-l-3}}(-2^{-l}e_n)$ if l is sufficiently large with respect to $1 - \tilde{h}_0, \alpha$ and ϵ such that

$$
c_0(1 - \tilde{h}_0)\epsilon\delta_l > c_0(1 - h_0)\epsilon l^{-2} > C2^{-\alpha l}.
$$

Thus if we choose $\epsilon = \epsilon(\theta_0, n) > 0$ small enough, then Lemma 1 in [C2] implies that

$$
p \cdot Du(x, t) \geq 0 \text{ in } B_{2^{-l-5}}(-2^{-l}e_n) \times [0, r_1 2^{-l}\delta_l],
$$

for $l > l_0(\theta_0, n)$, which proves our assertion. $\qquad \square$

Below we state a modified version of Lemma 9.3 in [CJK]. The proof is parallel to that in [CJK]. Combined with Lemma 6.1, the following lemma says that, for $(x_0, t_0) \in \Gamma(u)$, $\frac{Du}{|Du|}$ converges as we take smaller neighborhoods of (x_0, t_0). The rate of this convergence, in comparison with the size of the neighborhood, determines the regularity of $\Gamma(u)$ in space.

Lemma 6.2. *Let u solve (HS) in $B_1(0) \times [-\tau, \tau], 0 < \tau < 1$ with*

$$
(0, 0) \in \Gamma(u), t(e_n; u) = 1 \text{ and } |Du| > m_0 \text{ on } \Gamma(u).
$$

In addition suppose there exists a unit vector $\nu \in \mathbb{R}^n$ and $0 < b_0 < 1$ such that

$$
\alpha(Du, -e_n) \leq \delta \text{ in } B_1(0) \times (-\tau, \tau).
$$

with δ smaller than a dimensional constant, and

$$
\alpha(Du(x, t), \nu) \leq b_0\delta \text{ in } B_{1/16}(-e_n) \times (-\tau, \tau).
$$

Then there exists a unit vector $\nu_1 \in \mathbb{R}^n$ and a constant $0 < c < 1$ depending only on n, m_0 and b_0 such that

$$
\alpha(Du(x, t), \nu_1) \leq \delta_1 \text{ in } B_{1/2}(0) \times (-\tau/2, \tau/2)
$$

where $\delta_1 \leq \delta - c\delta\tau$.

Now we go back to our original solution u. Let us fix $P_0 \in \Gamma_0 \cap B_r(0)$

$s_0 \in [t(P_0 + se_n), t(P_0 + 2se_n)], 0 < s \le r$ and $x_0 \in \Gamma_{s_0}(u) \cap B_{3s}(P_0)$.

Change the coordinate and re-scale as follows:

$$\breve{u}(x,t) = \frac{\alpha}{s} u_2(sx + x_0, \alpha t + s_0), \alpha = t(se_n; u). \tag{6.4}$$

Next, we construct blow-up family $\{u_l\}$ of u which satisfy the hypothesis of Lemma 6.1 as follows:

$$u_{l+1}(x,t) := 4^l \alpha_l \breve{u}(2^{-l} x, \alpha_l t),$$

where $\alpha_0 = 1$ and for $l \ge 1$

$$\alpha_l = t(2^{-l} e_n; \breve{u}).$$

Note that u_l ($l = 1, 2 \ldots$) is a viscosity solutions of (HS) in $B_1(0) \times (-1, 1)$ with the property $t(e_n; u_{l+1}) = 1$. Also recall that $\Gamma(u)$ is Lipschitz with Lipschitz constant $L < L_n$ and by Corollary 5.3

$$|Du_l(x,t)| \ge m_0 \text{ for } (x,t) \in (B_1(0) \cap \Omega_t(u_l)) \times (-1/2, 1).$$

with $m_0 = m_0(n)$.

Now we apply an iteration argument, starting with sufficiently large l if necessary. Suppose that $u_l(\cdot, t)$ is monotone for the cone $W(\nu_l, \theta_l)$ in $B_2(0) \times (-\delta_l, \delta_l)$ where
$$\delta_l = \pi/2 - \theta_l > l^{-2}.$$

Then $\breve{u}$ satisfies the hypothesis of Lemma 6.1, and thus there exists a unit vector ν and $0 < r_0(n), h_0(n) < 1$ such that $u_{l+1}(\cdot, t)$ is monotone increasing in $B_{1/16}(-e_n) \times (-2r_0\delta_l, 2r_0\delta_l)$ for the cone $W(\theta_{l+1}, \nu)$ with $\delta_{l+1} \le h_0\delta_l$. Now Lemma 6.2 applies to u_{l+1} to yield the enlarged cone of monotonicity $W(\nu_{l+1}, \theta_{l+1})$ with

$$\pi/2 - \theta_{l+1} = \delta_{l+1} = \delta_l - cr_0\delta_l^2, \quad c = c(n) \tag{6.5}$$

for u_{l+1} in $B_{1/2}(0) \times [-r_0\delta_l, r_0\delta_l]$. Now we can repeat the process with δ_{l+k}, where $k = k(n)$ such that $2^{-k} < r_0$.

From (6.5) we obtain
$$\delta_l \approx \frac{C}{l},$$

which yields the differentiability of $\Gamma(u)$ in space at $(0,0)$:

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Theorem 6.3. *Let u and r be as given in Proposition 4.1, and let $\tau = t(re_n; u)$. Then $\Gamma(u)$ is C^1 in space in $B_r(0) \times [-\tau, \tau]$. In particular, there exist constants $l_0, C_0 > 0$ depending only on n such $(x_0, t_0) \in \Gamma(u)$,*

$$\Gamma_t(u) \cap B_{2^{-l}}(x_0) \text{ for } |t - t_0| \leq 2^{-l}$$

is a Lipschitz graph with respect to direction $\frac{Du}{|Du|}(-2^{-l}e_n, t_0)$, with Lipschitz constant less then $\dfrac{C_0}{l}$ if $l \geq l_0$.

7 Upper bound of the spatial gradient on the free boundary

To derive an upper bound of $|Du|$, we need stronger regularity than C^1 of $\Gamma(u)$ in space. For this we need a refined version of Lemma 6.1 as below:

Lemma 7.1. *Suppose u is as given in Lemma 6.1 with $l^{-2} \leq \delta_l \leq l^{-1}$. Then for $l \in \mathbb{N}, l > l_0$ where l_0 depending only on n and θ_0, there exist a unit vector $\nu_{l+1} \in \mathbb{R}^n$ and $0 < r_0, h_0, k < 1$ such that $u(\cdot, t)$ is monotone increasing in*

$$B_{2^{-l-5}/\sqrt{l}}\left(-\frac{2^{-l}}{\sqrt{l}}e_n\right) \times (0, r_0 2^{-l} l^k \delta_l)$$

along every direction $\eta \in W(\theta_1, \nu_1)$ with

$$\delta_{l+1} \leq h_0 \delta_l.$$

Here $r_0 = r_0(\theta_0, n)$ and k, h_0 only depends on θ_0 and n.

Proof. Proceeding as in the proof of Lemma 6.1, one can construct a star-shaped solution w of (HS) with $s = \frac{2^{-l}}{\sqrt{l}}$. Moreover arguing as before, $w(\cdot, t)$ is monotone for the cone $W(e_n, \theta'_l)$ in

$$B_{2^{-l+1}}(0) \times [0, t(2^{-l}e_n; w)],$$

where $\theta'_l = \theta_l - C2^{-\frac{\alpha}{14}l}$, $C = C(n)$. Observe that $\delta'_l = \delta_l + C2^{-\frac{\alpha}{14}l} \leq 2l^{-l}$ for sufficiently small l, and by Proposition 10.5 in [CJK] applied to w

$$w_t(x, t) = |Dw|^2(-(2^l\sqrt{l})^{-1}e_n, t)(1 + O(l^{-k}))$$

in the region

$$B_{\frac{1}{8}(2^l\sqrt{l})^{-1}}(-(2^l\sqrt{l})^{-1}e_n) \times [0, t(2^{-l}e_n; w)]$$

where $0 < k < 1/2$ and $C > 0$ depends only on n.

Due to Lemma 9.1 in [CJK] applied to w, there exists a unit vector $\nu_l \in \mathbb{R}^n$ and $0 < h_0 < 1$ such that

$$\alpha(Dw, \nu_l) \leq h_0 \delta_l \text{ in } B_{\frac{1}{8}(2^l\sqrt{l})^{-1}}(-(2^l\sqrt{l})^{-1}e_n) \times [0, r_0 2^{-l}l^k\delta_l].$$

Parallel argument as in the proof of Lemma 5.1 yields the conclusion. $\quad\square$

We can now apply an iteration argument as in section 6 to the family of functions
$$\tilde{u}^{l+1}(x, t) = 4^l l \beta_l \breve{u}(\frac{x}{2^l\sqrt{l}}, \beta_l t)$$

where $\breve{u}$ is defined in (6.4) and

$$\beta_l = t(\frac{2^{-l}}{\sqrt{l}}e_n; u)$$

Namely, $\{\tilde{u}^l\}$ is defined similarly as $\{u^l\}$ in section 6, but replacing the scaling factor 2^l by $2^l\sqrt{l}$. Then the corresponding δ_l obtained for $\tilde{u}_l$ satisfies

$$\delta_{l+1} = \delta_l - \delta_l^2 l^k,$$

and we obtain $\delta_l \sim l^{-\gamma}$, where $1 < \gamma < 2$. Thus the following theorem is obtained for $\breve{u}$:

Theorem 7.2. *There exist constants $l_0 > 0$ and $1 < \gamma < 2$ depending only on n such that for a free boundary point $(x_0, t_0) \in \Gamma(\tilde{u}) \cap (B_1(0) \times [0, 1])$, if $l > l_0$ then $\Gamma(\breve{u}) \cap B_{2^{-l}}(x_0, t_0)$ is a Lipschitz graph with Lipschitz constant less then $l^{-\gamma}$.*

Then by Theorem 2.4 in [W] we obtain the upper bound of $|D\breve{u}|$:

Corollary 7.3. *The spatial gradient $D\breve{u}(\cdot, t)$ exists in $\bar{\Omega}_t(\breve{u}) \cap B_1(0)$ for $0 \leq t \leq 1$ and*

$$|D\breve{u}(\cdot, t)| \leq C \text{ in } \Omega_t(u) \cap B_1(0) \text{ for } 0 \leq t \leq 1,$$

where C is a dimensional constant.

In terms of our original function u, we obtain the following statement:

Corollary 7.4. *Let u, r and τ as given in Theorem 6.3. Then for any $P_0 \in \Gamma_0 \cap B_r(0)$, $0 < t_0 = t(P_0 + se_n) < \tau$ we have*

$$C_1 \leq \frac{|Du(\cdot, t)|}{|Du(P_0 - se_n, t)|} \leq C_2 \text{ in } \Omega_t(u) \cap B_s(P_0 + se_n) \text{ for } \frac{t_0}{2} \leq t \leq t_0.$$

Here C_1 and C_2 are dimensional constants.

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8 Regularity in time

It remains to address the regularity of $\Gamma(u)$ in time. Here we will use the fact that, due to the second condition in (1.1), $e^{At}u$ is increasing in time, and therefore one-sided limits $u(x, t^-)$ and $u(x, t^+)$ exists for all (x, t).

Proposition 8.1. *Let u, r and τ be as in Theorem 6.3. Then $\Gamma(u)$ is Lipschitz continuous in time in $B_r(0) \times (0, \tau)$. More precisely, if $(x_0, t_0) \in \Gamma(u)$, then*

$$V^-_{(x_0, t_0)} := \lim_{(y,s) \in \Omega(u) \to (x_0, t_0), s < t_0} \frac{u_t}{|Du|}(y, s) = |Du|(x, t_0^-)$$

and

$$V^+_{(x_0, t_l)} := \lim_{(y,s) \in \Omega(u) \to (x_0, t_0), s > t_0} \frac{u_t}{|Du|}(y, s) = |Du|(x, t_0^+)$$

Remark Note that u may have jump discontinuities in time, and thus one cannot ensure that free boundary velocity is continuous in time.

Proof. 1. For $l = 1, 2, ...$ let us define

$$u^l(x, t) = 2^l u(x_0 + 2^{-l}x, t_0 + 2^{-l}t) \text{ in } B_{2^l}(0) \times [-2^l, 2^l].$$

Without loss of generality we may assume that $e_n := \frac{Du}{|Du|}(x_0, t_0)$. Theorem 6.3 yields that

$$\alpha(Du^l, e_k) \to 0 \text{ as } l \to \infty \text{ if } k \neq n, \tag{8.1}$$

locally uniformly in $\mathbb{R}^n$.

Moreover, due to Theorem 7.2 and Theorem 2.4 in [W],

$$|Du(\cdot, t) - Du(x_0 - 2^{-m}e_n, t)| \leq 1/m \text{ in } \Omega_t(u) \cap B_{2^{-m}}(x_0) \tag{8.2}$$

for $|t - t_0| \leq 2^{-m}$.

2. Let us we define, for a given domain $\Sigma \in \mathbb{R}^{n+1}$,

$$d_\Sigma(x, t) := \text{ signed distance function to } \Sigma.$$

Formally speaking,

$$\frac{u_t}{|Du|} = |Du| \text{ on } \Gamma(u)$$

in a small neighborhood of (x_0, t_0). In particular $d_{\Omega(u^l)}$ is uniformly Lipschitz in time with respect to l. Due to a barrier argument based on Corollary 7.3, one can show that $\Gamma(u^l)$ has bounded normal velocity. Therefore if we define

$$u^\infty(x, t) := \limsup_{l \to \infty, (y,s) \to (x,t)} u^l(y, s)$$

and

$$u_\infty(x, t) := \liminf_{l \to \infty, (y,s) \to (x,t)} u^l(y, s),$$

Then due to Arzeli-Ascola, $d_{\Omega(u^\infty)}$ and $d_{\Omega(u_\infty)}$ are given by a local uniform limit of respective subsequences of $\{d_{\Omega(u^l)}\}_{l \in \mathbb{N}}$.

3. Due to (8.1) and (8.2),

$$u^\infty(x, t) = a_1(t)(a_2(t) - x_n)_+, \text{ and } u_\infty(x, t) = b_1(t)(b_2(t) - x_n)_+.$$

where $a_2(t)$ and $b_2(t)$ is increasing, $a_2(0) = b_2(0) = 0$ and Lipschitz continuous in time. Also observe that, due to (1.1) $(1 + 2^{-l}At)u^l$ increases in time, and thus u^∞ and u_∞ increases in time, and in particular $a_1(t)$ and $b_1(t)$ has one-sided limits.

4. We prove that $a_1(0-) = b_1(0-)$. Suppose

$$|a_1(0-) - b_1(0-)| > \delta > 0.$$

By definition of $a_1(t), b_1(t)$, one can choose sufficiently large l_1, l_2 and x_l, y_l so that

$$|a_1(0-) - Du^{l_1}(x_l, s_l^1)|, |b_1(0-) - Du^{l_2}(y_l, s_l^2)| < \delta/4 \qquad (8.3)$$

with $x_l, y_l \in B_1(0)$ and $s_l^i \to 0^-$ as $l \to \infty$. On the other hand, due to (1.1), $u(-2^{-m}e_n, 0^-)$ exists for any m, and thus if $|s_l^1 - s_l^2|$ and $|s_l^1|, |s_l^2|$ is sufficiently small depending on m with $s_l^1, s_l^2 < 0$, then

$$\frac{|u(-2^{-m}e_n, s_l^1) - u(-2^{-m}e_n, s_l^2)|}{2^{-m}} < \delta/4. \qquad (8.4)$$

Now due to (8.1) and (8.2), if l and m is chosen sufficiently large,

$$|Du(-2^{-m}e_n, s_l^1) - Du^l(x_l, s_l^1)| < \delta/4 \qquad (8.5)$$

and

$$|Du(-2^{-m}e_n, s_l^2) - Du^l(y_1, s_l^2)| < \delta/4. \qquad (8.6)$$

Putting (8.3)-(8.6) together, one obtains $|a_1(0-) - b_1(0-)| < \delta$, a contradiction.

5. Parallel argument holds for other cases to obtain

$$a_1(0-) = b_1(0-) = |Du|(x_0, t_0-) \text{ and } a_1(0+) = b_1(0+) = |Du|(x_0, t_0+).$$

Now it is straightforward from barrier arguments, using the fact that $\{u^l\}$ are viscosity solutions of (HS), to prove that

$$a_2'(0-) = b_2'(0-) = |Du|(x_0, t_0-) \text{ and } a_2'(0+) = b_2'(0+) = |Du|(x_0, t_0+),$$

$\square$

9 Global solutions with initially Lipschitz free boundaries

As an application of Theorem 1.2, we will show existence, uniqueness and regularity and properties of global solutions of (HS). Let

$$\Omega_0 := \{x_n \leq g(x'), x = (x', x_n)\} \tag{9.1}$$

where g is Lipschitz continuous with Lipschitz constant $L < L_n$ and $g(0) = 0$. Then there is a unique harmonic function u_0 in Ω_0 such that $u_0(-e_n) = 1$ and $u_0 = 0$ on Γ_0.

Definition 9.1. *A lower semi-continuous function $u(x,t)$ in $\mathbb{R}^n \times [0, \infty)$ is a global viscosity solution of (HS) with initial data u_0 if*

(a) $u(x,t)$ is a viscosity solution of (HS) in $B_R(0) \times (0, \infty)$ for any $R > 0$,

(b) u increases in time, and

(c) u locally uniformly converges to u_0 as $t \to 0$.

Let u be a global viscosity solution with initial data u_0. To address uniqueness and regularity properties of u, we normalize the solution by assuming that

$$\lim_{R \to \infty} \frac{u(-Re_n, t)}{u_0(-Re_n)} = 1 \text{ for each } t \geq 0. \tag{9.2}$$

Lemma 9.2. *Suppose u is a global solution of (HS) with (9.1). Then for any $\epsilon > 0$ and $T > 0$ there exists $R_0 > 0$ such that if $R > R_0$*

$$d(x, \Gamma_0(u) \cap B_R(0)) \leq \epsilon R \text{ for any } x \in \Gamma_t(u), 0 \leq t \leq T. \tag{9.3}$$

Proof. Denote the following property by $(P)_{t,R}$:

$$(P)_{t,R} \qquad\qquad u \leq \epsilon R^2 \text{ on } \partial B_R(0) \times [0,t]$$

where $\epsilon > 0$ is a sufficiently small dimensional constant. Note that by (9.2) and the Harnack inequality, there exists $R_0 = R_0(t) > 0$ such that

$$u(x - Re_n, t) \leq 2u(x - Re_n, 0) \leq C\epsilon R^{2-\alpha} \qquad\qquad (9.4)$$

for $x \in \Gamma_0 \cap B_R(0)$ and $R \geq R_0$.

Now suppose that (9.3) is violated at $t = t_0$. This means that for any $\epsilon > 0$ there are sequences $R_k \to \infty$ with which $(P)_{t_0, R_k}$ fails. (Otherwise a barrier argument using radially symmetric barriers will yield (9.3).) In other words, at every $R_k = 2^k R_0(t_0)$ there exists $x_k \in B_{R_k}(0)$ such that $u(x_k, t_0) \geq \epsilon R_k^2$. Since u increases in time, we have

$$u(x_k, t) \geq \epsilon R_k^2 \text{ for } t \geq t_0.$$

Again from a barrier argument it follows that at $t_1 = t_0 + \frac{3}{\epsilon}$, the positive phase $\Omega_{t_1}(u)$ contains $B_{3R_k}(x_k)$ for any k. But then $B_{3R_k}(x_k)$ contains $x = -R_k e_n$. This and the Harnack inequality contradicts (9.4) if R_k is chosen sufficiently large so that $R_k > R_0(t_1)$ and $\epsilon >> R_k^{-\alpha}$. $\qquad\square$

We now state our main theorem:

Theorem 9.3. *There exists a unique global solution u of (HS) with initial data u_0 which satisfies (9.1). Moreover the following properties hold for u:*

(a) *For any $t > 0$, $\Gamma_t(u)$ is a Lipschitz graph with respect to the direction e_n with Lipschitz constant L in $\mathbb{R}^n$.*

(b) *$\Gamma(u) \cap \{t > 0\}$ is smooth in space and time.*

(c) *For any $x \in \Gamma_0(u)$ such that $x + de_n \in \Gamma_t(u)$,*

$$C^{-1} \leq \frac{|Du(x + de_n, t)|}{|Du(x - de_n, t)|} \leq C \qquad\qquad (9.5)$$

for a dimensional constant $C > 0$.

In addition, if $g(-x') = g(x')$, then $u(x', x_n, t) = u(-x', x_n, t)$ and the Lipschitz constant of the graph $\Gamma_t(u)$ in any bounded neighborhood in space decreases in time. In particular the Lipschitz constant of $\Gamma_t(u)$ at time $t(de_n; u)$ in $\{x : |x_n| \leq 2^{-l}d\}$ is less than $\min[L, \frac{C}{l^\gamma}]$ with $1 < \gamma < 2$.

Before proving Theorem 9.3 we first apply the theorem to show that the local solution u given in Theorem 1.2 has a unique blow-up profile at $t = 0$ if the subsequence is chosen such that the blow-up limit of the initial data is locally uniform. Let us define

$$u_k(x,t) := \alpha_k u(\frac{x}{k}, \beta_k t), \quad \alpha_k = u(-\frac{e_n}{k}, 0)^{-1}, \beta_k = \frac{\alpha_k}{k^2},$$

where u is as given in Theorem 1.2. Note that by definition of viscosity solutions for any $x \in I\!R^n$

$$|u(x,t) - u(x,0)| \to 0 \text{ as } t \to 0. \tag{9.6}$$

Also observe that, since L_n is chosen such that u has less than quadratic decay near the free boundary,

$$\beta_k = (k^2 u(-\frac{e_n}{k}, 0))^{-1} \to 0 \text{ as } k \to 0.$$

Therefore for any $t > 0$ and sufficiently small d

$$|\frac{u_k(-dke_n, t)}{u_k(-dke_n, 0)} - 1| = \frac{|u(-de_n, \beta_k t) - u(-e_n, 0)|}{u(-de_n, 0)} \to 0 \text{ as } k \to \infty. \tag{9.7}$$

Now the second equation Corollary 5.7 applied to u yields that $\Gamma_t(u_k)$ is at most t^γ away from $\Gamma_0(u_k)$ in $B_k(0)$, where $0 < \gamma < 1$. Since $\Gamma_t(u)$ is Lipschitz continuous in space, it follows from (9.7) with Lemma 2.13 that

$$|\frac{u_k(-Re_n, t)}{u_k(-Re_n, 0)} - \frac{u_k(-dke_n, t)}{u_k(-dke_n, 0)}| \leq \frac{t^\gamma}{R} + d^\gamma.$$

if $R < dk$.

Hence for any $\epsilon > 0$ there exists $R > 0$ such that

$$\limsup_{k \to \infty} |\frac{u_k(-Re_n, t)}{u_k(-Re_n, 0)} - 1| \leq \epsilon. \tag{9.8}$$

Due to Theorem 1.2 (a), (c) and Corollary 5.7 it follows that $\{u_k\}$ is locally uniformly Hölder continuous in space and $\Gamma(u_k)$ is locally uniformly Lipschitz continuous in space and Hölder continuous in time. Hence by Ascoli-Arzela, along a subsequence $u_k(\cdot, t)$ and $d_{\Omega(u_k)}$ locally uniformly converges to $u^\infty(\cdot, t)$ and d_{Ω^∞}. It follows that

$$(u^\infty)^*(x,t) = \limsup_{0 \leq s \to t, y \to x} u^\infty(y, s)$$

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and

$$(u^\infty)_*(x,t) = \liminf_{0 \le s \to t, y \to x} u^\infty(y,s).$$

It follows from the stability properties of viscosity solutions that $(u^\infty)_*$ is a global solution of (HS). (9.1) follows from (9.8) and the local uniforml convergence of u_k to u^∞. Lastly u^∞ increases in time due to the second condition of (1.1). Due to Theorem 9.3 we have the following:

Theorem 9.4. u^∞ *is the unique global viscosity solution of (HS) with initial positive phase*

$$\Omega_0^\infty := \{u^\infty(\cdot, 0) > 0\}$$

satisfying (9.1).

Note that Ω_0^∞ is a subsequential limit of $\{u_k(\cdot, 0) > 0\}$, i.e., a subsequential limit of the blow-up profile of Ω_0 at the origin.

Proof of Theorem 9.3.

1. Let Ω_0 be as given in (9.1). First to show the existence, we consider a sequence u_N of star-shaped and Lipschitz initial positive phase Ω^N which coincides with Ω_0 in $B_{N^2}(0)$, and with fixed boundary data m_N on $K_N = B_1(-Ne_n)$ such that $u_N(-e_n, 0) = 1$. Then due to the main theorem in [CJK], $\{u_N\}_N$ is locally uniformly Hölder continuous in space and time, and thus converges locally uniformly to u along a subsequence. It then follows from the construction and the uniform Hölder continuity of $\{u_N\}$ in time that u satisfies (9.1) and (9.2). In addition u is increasing in time and thus a global viscosity solution of (HS).

2. Next we show the regularity of u. For given $R > 0$, let us define the re-scaled function

$$u_R(x,t) := C(R)R^{-1}u(Rx, C(R)t),$$

where $C(R) = \frac{R}{u(-Re_n, 0)}$. Due to (9.2) it follows that, for any given $T > 0$ there exists R_0 such that

$$u_R(-e_n, t) \in [1, 2] \text{ for } 0 \le t \le \frac{T}{C(R)}, \text{ if } R > R_0. \tag{9.9}$$

Since u is increasing in time, so is u_R.

Moreover due to (9.3), for given $T > 0$ there exists $R_0 > 0$ such that if $R > R_0$

$$\Omega_{T/C(R)}(u_R) \cap B_1(0) \subset \Omega_0 + c_0 e_n, \tag{9.10}$$

where c_0 is as given in Theorem 1.2.

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Recall that $\Gamma_0(u_R)$ is Lipschitz continuous in $\mathbb{R}^n$. with small Lipschitz constant $L < L_n$. Due to (9.9) and (9.10), Theorem 1.2 and Corollary 5.6 apply to u_R to yield that $\Gamma_t(u_R)$ is smooth, (9.5) holds, and is a Lipschitz graph with Lipschitz constant $L + r^\alpha$ in $B_r(0)$ for

$$0 \le t \le \min(\frac{T}{C(R)}, t(re_n; u_R))$$

for any $r < r(n)$, where $0 < \alpha < 1$ depends on n.

In terms of u, this means that $\Gamma_t(u)$ is a Lipschitz graph with Lipschitz constant $L + r^\alpha$ in $B_{rR}(0)$ for $0 \le t \le \min(T, t(rRe_n; u))$.

Note that, due to Corollary 5.7 and the fact that $L < L_n$,

$$t(rRe_n; u) > c\frac{(rR)^2}{Cu(-rRe_n, 0)} \ge c(rR)^2$$

where c is a dimensional constant. Hence if we choose

$$r = R^{-1/(1+\alpha/2)} \text{ and } R = T,$$

then it follows that $\Gamma_t(u)$ is a Lipschitz graph with Lipschitz constant $L + R^{-\alpha/2}$ in $B_{R^{2\alpha}}(0)$. Since R can be chosen arbitrarily large, we conclude.

3. Now we show the uniqueness of u. Suppose u and v are two global solutions satisfying (9.2)-(9.3). For given R and $x \in \mathbb{R}^n$, let $\Psi_R(x) := \varphi^{-1} \circ \phi \circ \varphi(x)$, where $\varphi(x) = \frac{\epsilon}{R}x$ and $\phi = \psi(x + e_n) - e_n$, where

$$\psi(x) = \frac{1}{|x|^2} \cdot (x_1, ..., x_{n-1}, -x_n + 2|x|^2),$$

where $x = (x_1, ..., x_n) \in \mathbb{R}^n$. Since ψ is the composition of reflection about $x_n = 1$ and the conformal mapping sending x to $\frac{x}{|x|^2}$, Ψ_R is conformal. (Ψ_R fixes 0 and will bend $\Gamma(u)$ in the direction of e_n by ϵR on $\partial B_R(0)$). We compare $\tilde{u}$ and v in $B_R(0) \times [0, T]$, where

$$\tilde{u}(x, t) = (1 + 2\epsilon^{1/2})u((\Psi_R)^{-1}(x), (1 + 3\epsilon^{1/2})t)$$

Since $1 - \epsilon \le |D\Psi_R| \le 1 + \epsilon$ in $B_R(0)$, $\tilde{u}$ is a supersolution of (HS) in $B_R(0)$.

Due to (9.3),

$$\Omega(v) \subset \Omega(u) \text{ in } (B_{\epsilon^{-1/2}R}(0) - B_R(0)) \times [0, T]$$

if R is chosen sufficiently large with respect to ϵ and T. Moreover for sufficiently large R, (9.2) and the construction of $\tilde{u}$ yields that

$$v(-Re_n, t) \le (1 - \epsilon^{1/2})\tilde{u}(-Re_n, t).$$

Recall that both $\Gamma(u)$ and $\Gamma(u)$ are Lipschitz in space, due to our previous argument. Therefore Lemma 2.10 yields that $v \leq \tilde{u}$ in $\partial B_R(0)$ as long as $\Omega(v) \subset \Omega(u)$ in $B_R(0)$.

Now Theorem 2.7 yields

$$v \prec \tilde{u} \text{ in } B_R(0) \times [0, T].$$

Since ϵ can be chosen arbitrarily small and $R \to \infty$ as $\epsilon \to 0$, we conclude that $v \leq u$. Similarly $u \leq v$.

4. Finally the last statement of our theorem follows from the uniqueness of global solutions, Theorem 7.2 and the fact that Du is always parallel to e_n on the line $\{|x'| = 0\}$.

$\square$

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