Math 251A Spring 2024: Homework 4. Due June 7th.

1. Recall that, for any $u \in BV(U)$, there is a Radon measure μ and a μ-measurable function $\sigma : U \to I\!R^d$ with $|\sigma| \leq 1$ such that

$$\int_U u \nabla \cdot \phi dx = -\int_U \phi \cdot \sigma d\mu$$

for all $\phi \in C_c^1(U; I\!R^d)$. For a bounded domain U with C^1 boundary, show that if $u_k \in C^\infty(U) \cap BV(U)$ such that u_k converges to u in $L^1(U)$ and $\int_U |Du_k| \to \int_U |Du|$, then we have

$$\int_U Du_k \cdot \phi dx \to \int_U \phi \cdot \sigma d\mu$$

for all $\phi \in C_c^1(U; I\!R^d)$. (Namely Du_k converges to $\sigma d\mu$ as measures, though this statement requires showing the optional exercise below).

[Optional] Show that one can obtain the convergence for all $\phi \in C_c(U, I\!R^d)$.

2. Let us denote $B_r := \{|x| \leq r\}$ in $I\!R^d$. Show that, if E minimizes $Per(A)$ among measurable sets A with $A = L$ in B_1^c, and if $0 \in \partial E$, then

$$\min[|E \cap B_r|, \ |E^c \cap B_r|] \geq c_d r^d \quad \text{for all } 0 < r < 1,$$

where c_d is a dimensional constant. The proof presented in class is not completely rigorous as we discussed in class, since it assumes that
$Per(E \setminus B_r, B_1) = Per(E, B_1 \setminus B_r) + H^{d-1}(E \cap \partial B_r)$.

3. Using the isoperimetric inequality, show that

$$\|u\|_{L^{d/(d-1)}(I\!R^d)} \leq C\|Du\|_{L^1(I\!R^d)} \text{ for any compactly supported } u \in BV(I\!R^d).$$

Please characterize the optimal constant C.

4. Consider $U \in I\!R^n$ which is bounded and with smooth boundary, and consider minimizing $P(F, U)$ among sets in $M := \{F : \chi_F \in BV(U), |F \cap U| = 1\}$. Show that

(a) There is a minimizer E in M. Is E always unique? (provide a counterexample for $n \geq 2$).

(b) Show that, under the assumption that $\partial E \cap U$ is locally a C^2 graph, its mean curvature is globally constant.

(c) [Optional] Suppose now that in a small neighborhood $\mathcal{N}$ of $x_0 \in \partial E \cap \partial U$, ∂E can be presented as a C^2 graph. Show that ∂E meets ∂U orthogonally, i.e. the outward normal ν of E at x_0 is tangent to ∂U.

5. [Saddle points] Find a compact set S which consists of two sets S_1 and S_2, each with smooth boundary, such that it is a critical point of $Per(E; I\!R^d)$ among sets of volume 1, but it is not a local minimizer of the energy (among sets of volume 1). Namely, show that

(a) The mean curvature is constant on ∂S (first variation is zero);

(b) There is a family of sets E_n with volume 1 with $|E_n \Delta S| \to 0$ such that

$$Per(E_n; I\!R^d) < Per(S; I\!R^d).$$

6. Let B be a unit ball in $I\!R^d$, and let $X = H^{-1}(B)$ denote the dual space of $H_0^1(B)$ with the associated norm. For the energy $E : X \to I\!R \cup \{+\infty\}$ defined by $E(\rho) = \int_B |D\rho|^2 dx$ if $\rho \in H^1(B)$, otherwise $+\infty$, derive formally the PDE representing the gradient flow of E in X.

Here, similar to the first example we discussed in class, we will define the tangent space at any point as $H^{-1}(B)$, with the inner product defined by

$$< u_1, u_2 >_{H^{-1}(B)} := \int_B f_1 f_2 dx,$$

where $f_i \in H_0^1(B)$ solves $-\Delta f_i = u_i$. Perturbation of a function $u \in H^{-1}(B)$ by ϕ in the tangent space is by addition, namely $u + \epsilon\phi$, the same as in first variation and the first example in class. This structure makes the time derivative the same as the standard one.