Math 251A Spring 2024: Homework 3. Due: May 20th

1. Let U be a domain in $\mathbb{R}^d$, and suppose that $u \in L^1(U)$ satisfies

$$\int_U u\Delta\phi = 0$$

for all smooth functions ϕ with compact support in U. Show that there is w which is equal to u a.e. such that w is harmonic in U.

2. [Harnack inequality] Let B_r denote $B_r(0)$ and define $\text{osc}_{B_r} u := \sup_{B_r} u - \inf_{B_r} u$.

(a) Show that $\sup_{B_{1/2}} u \le C_n \inf_{B_{1/2}} u$ if u is a positive harmonic function in B_1.

(b) Using (a), show the oscillation decay of harmonic functions:

$$\text{osc}_{B_{1/2}} u \le \frac{C_n - 1}{C_n + 1} \text{osc}_{B_1} u, \tag{0.1}$$

where u is harmonic but not necessarily positive.

(c) explain how (b) yields that u is Hölder continuous in $\bar{B}_{1/2}$.

3. Show that for any $\epsilon > 0$ there is a constant $C_\epsilon > 0$ such that

$$\|Du\|_{L^\infty(B_1)} \le \epsilon [Du]_{C^{0,\alpha}} + C_\epsilon \|u\|_{L^\infty(B_1)} \text{ for any } u \in C^{1,\alpha}(B_1).$$

4. Let $u \in C^1(B_1)$ be a H^1-solution to $-\nabla \cdot (A(x)\nabla u) = f$ with $f \in L^d(B_1)$ and continuous a_{ij} with ellipticity constants λ, Λ. Show that for any $0 < \alpha < 1$ we have

$$\|u\|_{C^\alpha(B_{1/2})} \le C(\|u\|_{L^\infty(B_1)} + \|f\|_{L^d(B_1)}),$$

where C only depends on $\alpha, d, \lambda, \Lambda$ and the mode of continuity for a_{ij}. Explain why the proof does not apply to the case to $\alpha = 1$.

5. [Boundary regularity] For a given $\alpha \in (0,1)$, let $u \in C^1(B_1) \cap C^\alpha(\overline{B_1})$ be a H^1 solution to $-\nabla \cdot (A(x)\nabla u) = 0$ in B_1, with continuous a_{ij} in $\bar{B}_1$ with ellipticity constants λ and Λ, and boundary data $g \in C^\alpha(\partial B_1)$. Show that

$$\|u\|_{C^\alpha(\overline{B_1})} \le C\|g\|_{C^\alpha(\partial B_1)},$$

where C depends on the same terms as in problem 4.

$\star$ For problem 4-5, one can use the following lemma which will be proved in class:

Lemma 0.1. *Let u and f be as given in Problem 4. Then*

$$\|\nabla u\|_{L^2(B_{1/2})} \leq C(\|u\|_{L^2(B_1)} + \|f\|_{L^2(B_1)}),$$

where $C = C(d, \lambda, \Lambda)$.

6. [Existence via regularity]

We will show that there exists at least one solution $u \in C^2(B_1) \cap C(\overline{B_1})$ of the following problem

$$\Delta u = \sin u \text{ in } B_1, \quad u = g \text{ in } \partial B_1,$$

with $g \in C^1(\partial B_1)$. Consider the map $T : X \to X$ with $X = C^{1/2}(\overline{B_1})$, where $T(w) = u$ with u being the unique solution of

$$\Delta u = \sin w \text{ in } B_1, \quad u = g \text{ in } \partial B_1.$$

(a) Show that $T(X)$ is contained in a compact subset of X.

(b) Apply the Schauder fixed point theorem to show that T has a fixed point.