Weak Separation, Pure Domains and Cluster Distance

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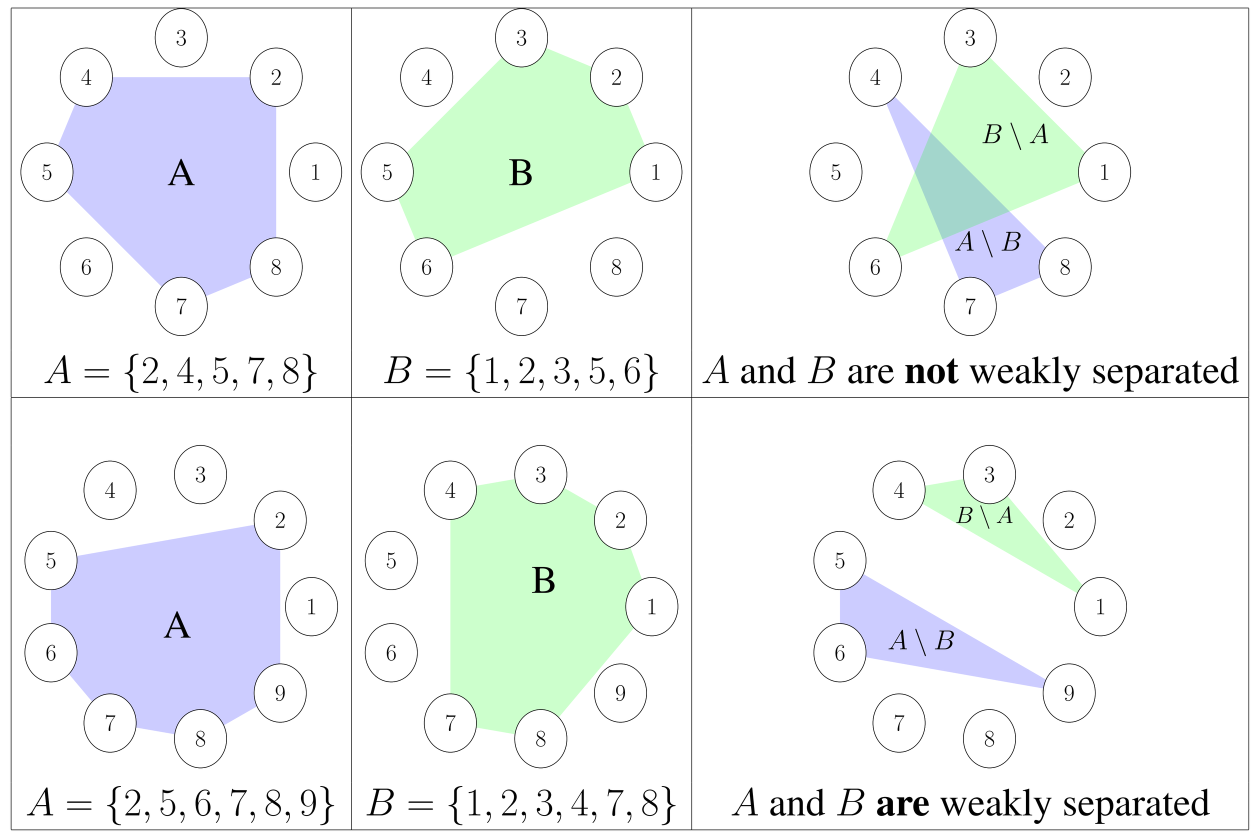
Weak separation

We denote by $\binom{[n]}{m}$ the collection of all m-element subsets of $[n] = \{1, 2, \dots, n\}$.

Definition. Two subsets $A, B \in \binom{[n]}{m}$ are *weakly separated* if the convex hull of $A \setminus B$ does not intersect the convex hull of $B \setminus A$ when drawn on a circle.



A collection of sets in $\binom{[n]}{m}$ is called *weakly separated* if any two of its elements are weakly separated from each other.

In their study of quasicommuting quantum minors, Leclerc and Zelevinsky stated the following *purity conjecture*:

Conjecture (Leclerc-Zelevinsky (1998))**.** *Every maximal (by inclusion) weakly separated collection in $\binom{[n]}{m}$ has size*

$$m(n - m) + 1.$$

For example, when $m = 2$, the maximal by inclusion weakly separated collections in $\binom{[n]}{2}$ are precisely triangulations of the n-gon. Each of them has $n - 3$ diagonals and n sides which agrees with the formula above: $2(n - 2) + 1 = 2n - 3$.

This conjecture has been confirmed independently by Danilov-Karzanov-Koshevoy in 2010 and by Oh-Postnikov-Speyer in 2015. Oh, Postnikov and Speyer showed that each such collection corresponds to a *plabic tiling*, which is a certain two-dimensional complex embedded in the plane. The dual objects to plabic tilings are *plabic graphs* that were introduced by Postnikov in 2006 while studying the totally nonnegative Grassmannian.

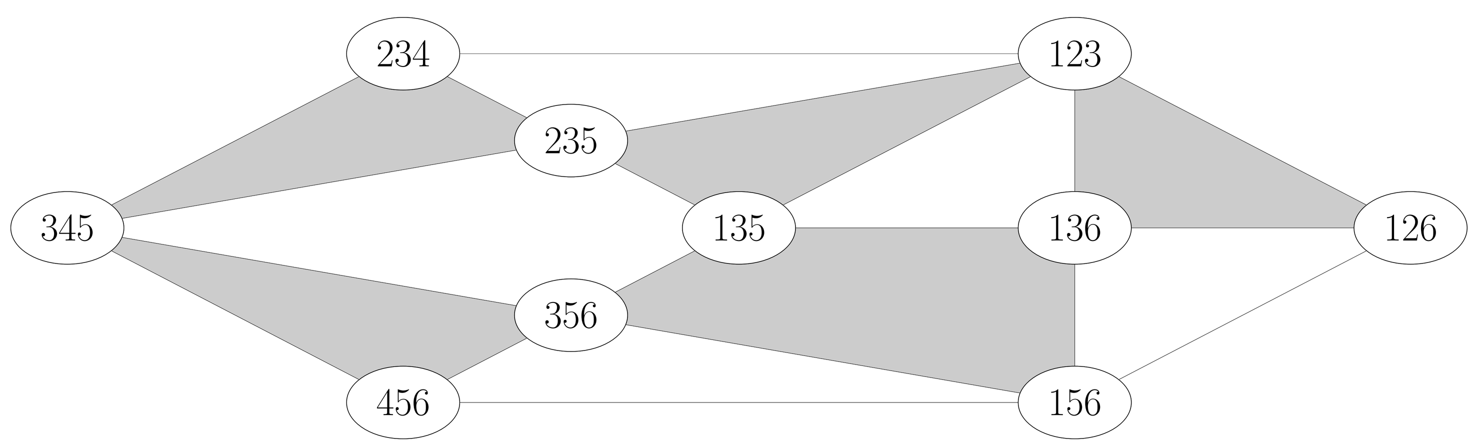


Figure 1: A plabic tiling corresponding to a maximal by inclusion collection of subsets in $\binom{[6]}{3}$. It has $10 = 3(6-3)+1$ vertices, and the collection consists of the sets written in their labels.

We say that a collection $\mathcal{A} \subset \binom{[n]}{m}$ is *a pure domain of rank* d if every maximal by inclusion collection of sets from $\mathcal{A}$ has size d, and in this case we write $\operatorname{rk}(\mathcal{A}) = d$. Thus the above conjecture can be restated as **the collection** $\binom{[n]}{m}$ **is a pure domain of rank** $m(n - m) + 1$.

Cluster distance

Given two subsets $I, J \in \binom{[n]}{m}$ that are not weakly separated, one can ask how close they are to being weakly separated. More specifically, let us denote $\mathcal{A}_{I,J} \subset \binom{[n]}{m}$ the collection of all sets weakly separated from both I and J.

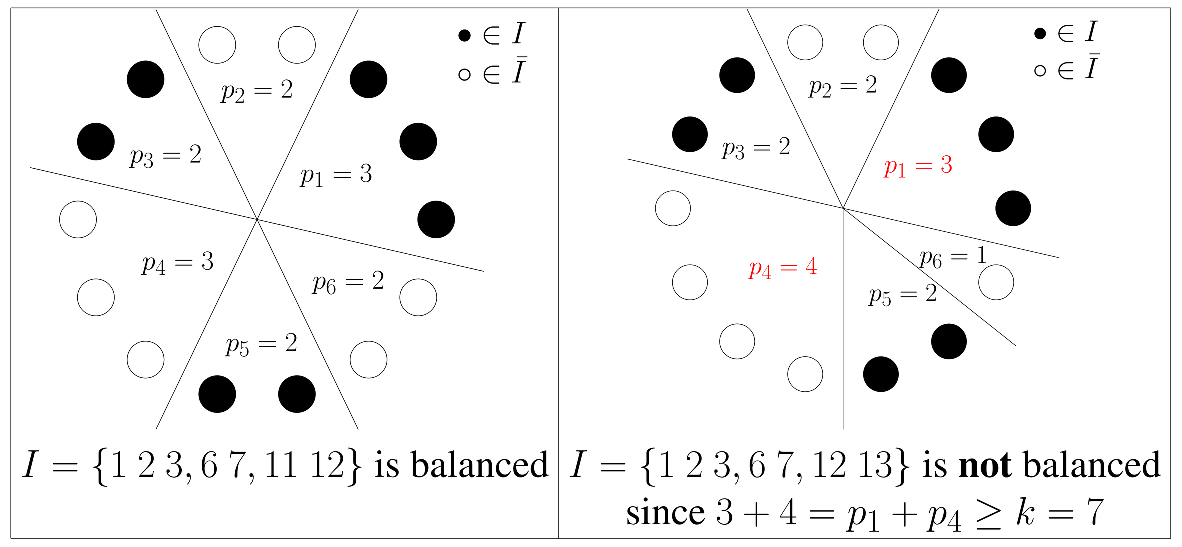
Definition (Farber-Postnikov (2015))**.** The *cluster distance* between I and J is defined by

$$d(I, J) = \operatorname{rk}\binom{[n]}{m} - \max\{\#\mathcal{C} \mid \mathcal{C} \subset \mathcal{A}_{I,J} \text{ is a weakly separated collection}\}.$$

Remark. Weakly separated collections correspond to clusters in the cluster algebra associated with the coordinate ring of the Grassmannian, and the notion of cluster distance between cluster variables generalizes in a straightforward way to all cluster algebras.

It turns out that for a lot of pairs (I, J), the domain $\mathcal{A}_{I,J}$ is pure and there is a simple formula for its rank. For example, if $m = 2$ and I and J are two crossing diagonals, $\mathcal{A}_{I,J}$ is pure of rank $2n - 4$. But first we concentrate on the case when I and J are complementary: $I \in \binom{[2k]}{k}$ and $J = \bar{I} = [2k] \setminus I$. In this case, I and its complement $\bar{I}$ partition the circle with $2k$ elements into an even number of cyclic intervals, denote their lengths $(p_1, p_2, \dots, p_{2r})$.

Definition. The set $I \in \binom{[2k]}{k}$ is called *balanced* if for all $i \neq j \in [2r]$, $p_i + p_j < k$.



Remark. As $k \to \infty$, the probability that a random set is balanced tends to 1. Thus "balanced" can be seen as an analogue of "generic".

Theorem. *If I is balanced, then $\mathcal{A}_{I,\bar{I}}$ is a pure domain of rank*

$$2k + \sum_{i=1}^{2r} \binom{p_i}{2}.$$

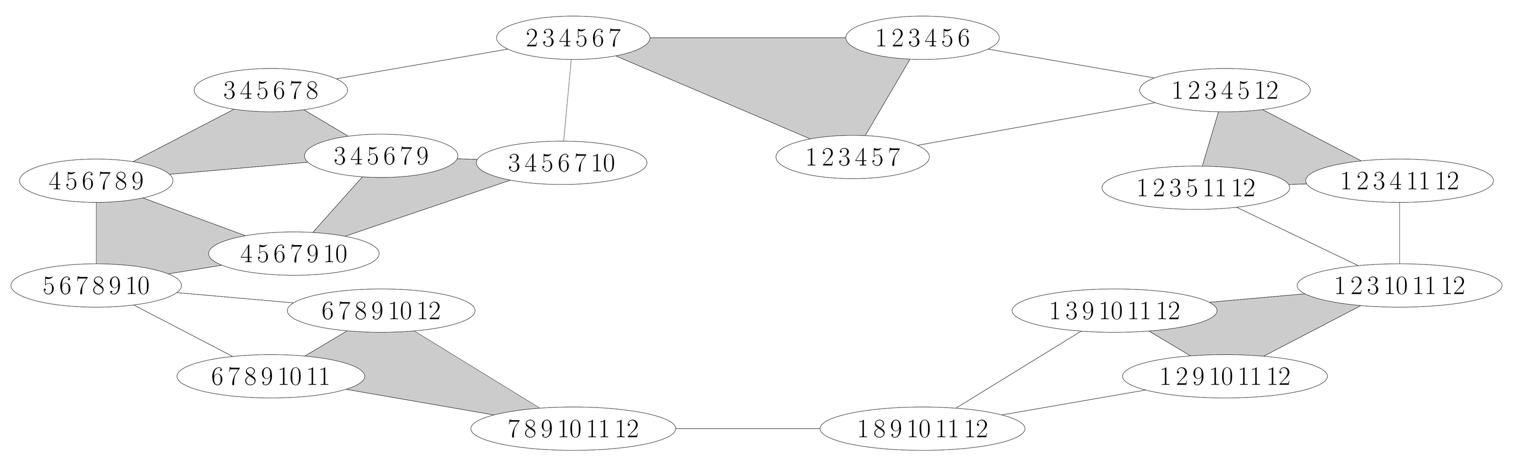


Figure 2: A maximal by inclusion weakly separated collection in $\mathcal{A}_{I,\bar{I}}$ for $I = \{1, 4\,5, 8\,9\,10\} \in \binom{[12]}{6}$. We have $(p_1, \dots, p_6) = (1, 2, 2, 2, 3, 2)$. Since I is balanced, the size of this collection equals $12 + 0 + 1 + 1 + 1 + 3 + 1 = 19$.

In the course of the proof, we show that the corresponding "partial" plabic tiling always fills in the domain between two simple closed curves. The outside curve is fixed and contains just all cyclic intervals, while the inside curve depends on the collection. We extensively use the results of both Oh-Postnikov-Speyer and Danilov-Karzanov-Koshevoy.

The non-complementary case

Let $I, J \in \binom{[n]}{m}$, and let $\pi(I), \pi(J) \in \binom{[2k]}{k}$ be the complementary subsets obtained from I and J by ignoring all the "irrelevant elements", i.e. the ones from $I \cap J$ and from $\overline{I \cup J}$. After that, $\pi(I)$ and $\pi(J)$ again partition the circle into an even number of cyclic intervals, and let $(p_1, p_2, \dots, p_{2r})$ denote their lengths.

Definition. We say that I and J form a *balanced pair* if $\pi(I)$ is balanced (equivalently, if $\pi(J)$ is balanced).

Theorem. *If I and J form a balanced pair, then $\mathcal{A}_{I,J}$ is a pure domain of rank*

$$\operatorname{rk}\binom{[n]}{m} - \operatorname{rk}\binom{[2k]}{k} + 2k + \sum_{i=1}^{2r} \binom{p_i}{2}.$$

Note that if $m = 2$ then two crossing diagonals do **not** form a balanced pair, however, the theorem still holds for this case, since we have $m = k = 2$ and $\binom{p_i}{2} = 0$ for all i.

We can rewrite $\operatorname{rk}\binom{[n]}{m} - \operatorname{rk}\binom{[2k]}{k}$ as $m(n - m) - k^2$. Or, in terms of cluster distance,

$$d(I, J) = 1 + k^2 - 2k - \sum_{i=1}^{2r} \binom{p_i}{2}.$$

This number does not depend on n and m. For the unbalanced case, we show that the same value gives an upper bound on the cluster distance:

Theorem. *For any $I, J \in \binom{[n]}{m}$,*

$$d(I, J) \leq 1 + k^2 - 2k - \sum_{i=1}^{2r} \binom{p_i}{2}.$$

Left-Right purity

Definition. For a positive integer n, denote by $\mathcal{LR}([0, n])$ the collection of all subsets $I \subset [0, n] = \{0, 1, \dots, n\}$ such that I contains **exactly one** of the elements $0, n$.

The following seemingly unrelated instance of the purity phenomenon is an important ingredient of our proof:

Theorem. *The collection $\mathcal{LR}([0, n])$ is a pure domain of rank*

$$\binom{n}{2} + n + 1.$$

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