Critical varieties in the Grassmannian

Pavel Galashin (UCLA)

FPSAC 2021
January 20, 2022

[GP18] (with P. Pylyavskyy) "Ising model and the positive orthogonal Grassmannian." *Duke Math. J.* arXiv:1807.03282

[Gal20] "A formula for boundary correlations of the critical Ising model." *Probab. Theory Related Fields.* arXiv:2010.13345

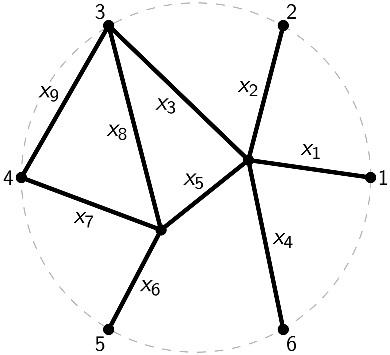
[Gal21a] "Critical varieties in the Grassmannian." arXiv:2102.13339.

[Gal21b] "Totally nonnegative critical varieties." arXiv:2110.08548.

[Gal21c] "Poset associahedra." arXiv:2110.07257.

Ising model

- $(G, \mathbf{x})$: weighted planar graph embedded in a disk.
- $0 < x_e < 1$ for all $e \in E(G)$



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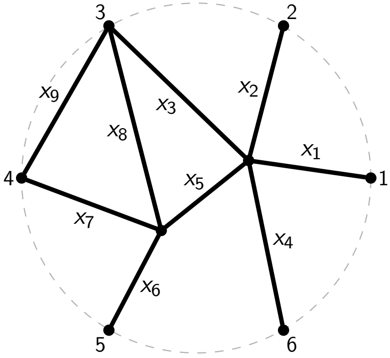

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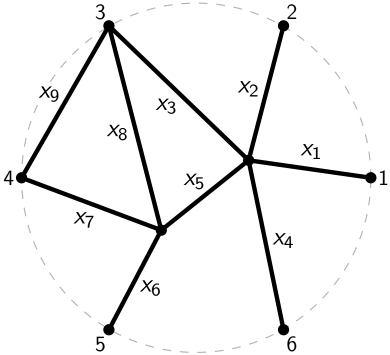

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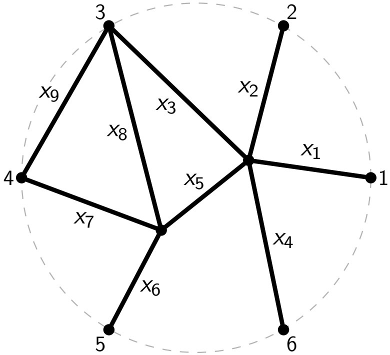

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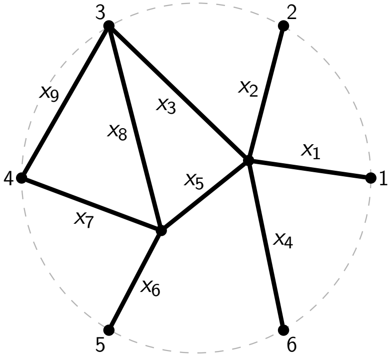

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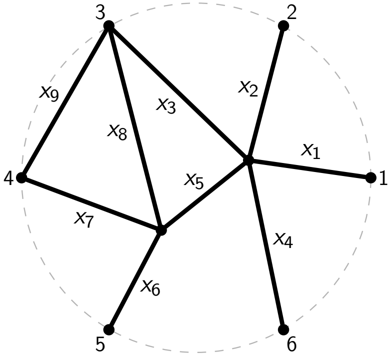

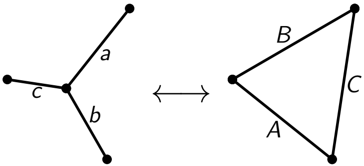

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$$A = \sqrt{\frac{(abc+1)(a+bc)}{(b+ac)(c+ab)}};$$

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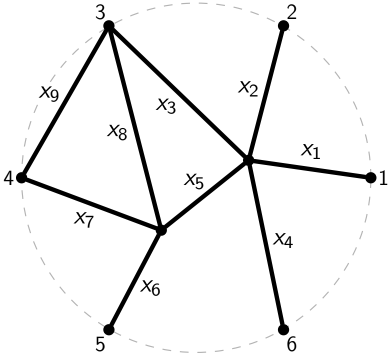
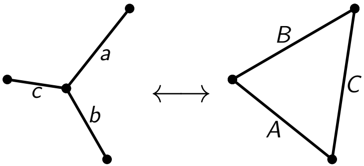


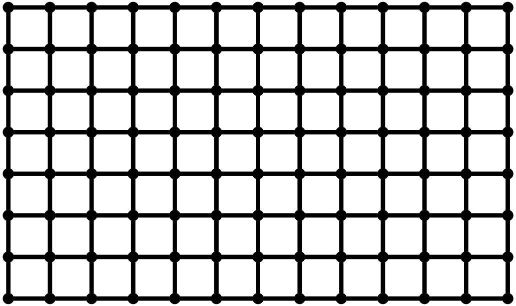


Phase transition

$$\text{Prob}(\sigma) \propto \prod_{\sigma_u \neq \sigma_v} x_{\{u,v\}}.$$

Usually:

- G = large piece of a (e.g. square) lattice;
- $x_e = x$ for all $e \in E(G)$.



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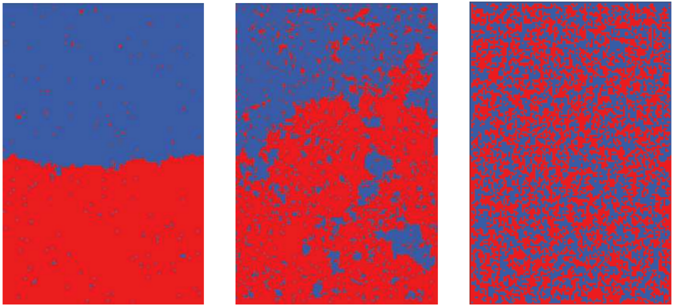
- Get a phase transition at critical temperature x_{crit}.

red=+ spin blue=− spin



$x < x_{\text{crit}}$ $x = x_{\text{crit}}$ $x > x_{\text{crit}}$

Picture credit: Dmitry Chelkak

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- Square lattice: $x_{\text{crit}} = \sqrt{2} - 1$.

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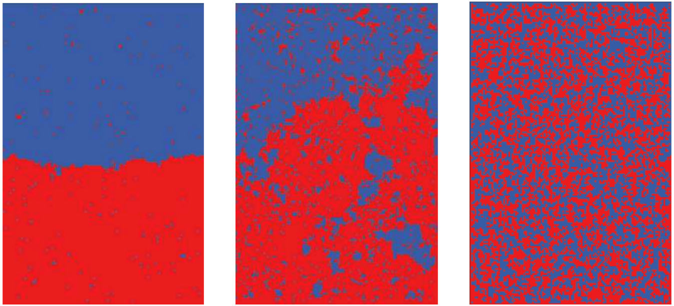

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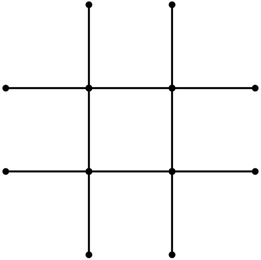

Square lattice

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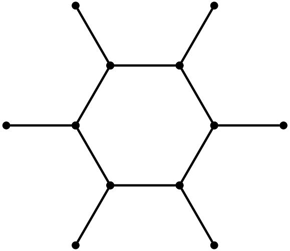
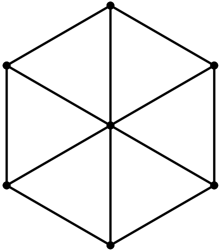


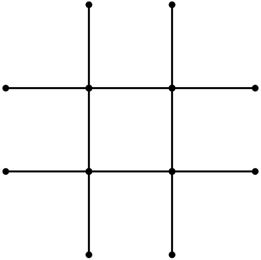
Hexagonal lattice

$x_{\text{crit}} = 2 - \sqrt{3}$



Triangular lattice

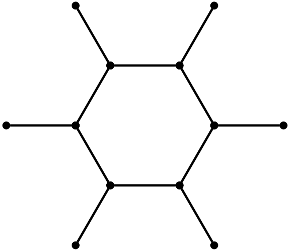
$x_{\text{crit}} = \frac{1}{\sqrt{3}}$



Square lattice

$x_{\text{crit}} = \sqrt{2} - 1$

$x_{\text{crit}} = \tan(\pi/8)$



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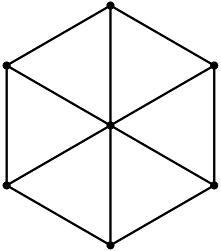

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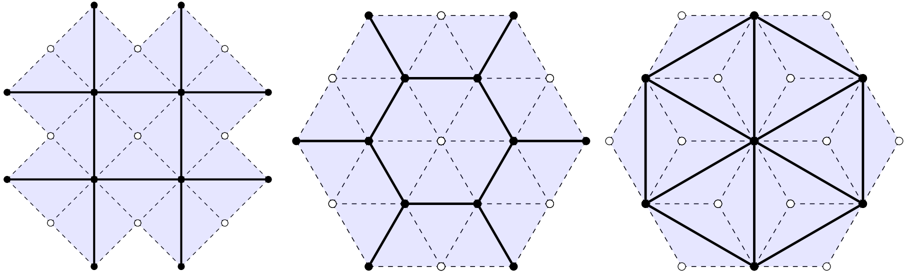

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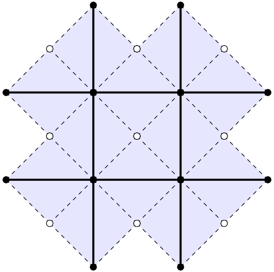

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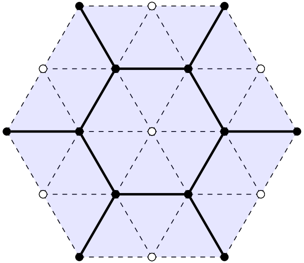

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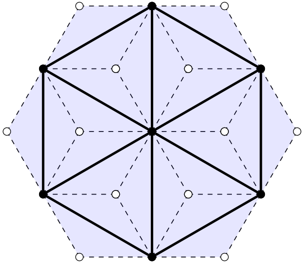

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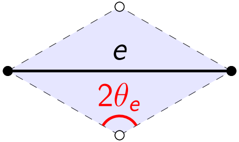

$x_e = \tan(\theta_e/2)$

Critical Z-invariant Ising model

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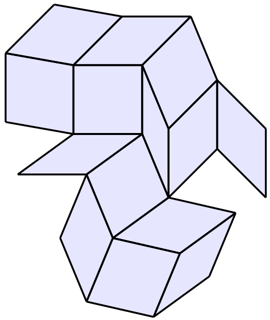

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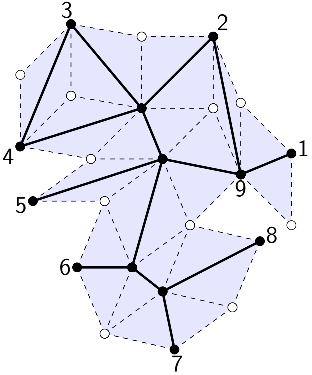

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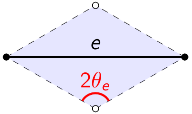

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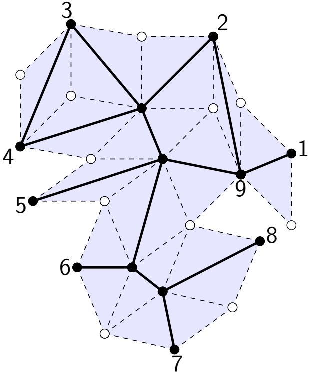

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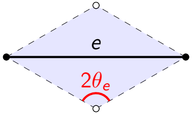

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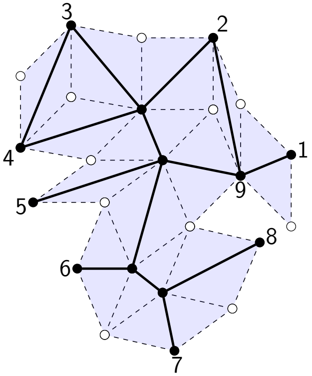

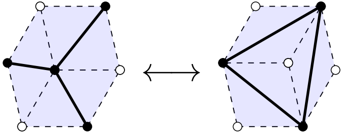

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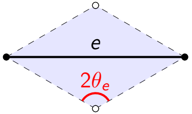

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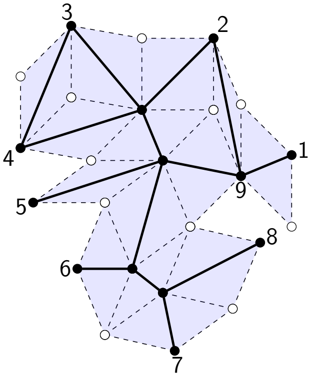
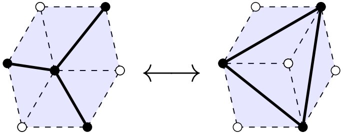
- Z-invariance: these edge weights are invariant under star-triangle moves.
- Conclusion: boundary correlation matrix M_R depends only on the shape of the region R.





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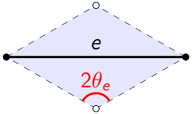

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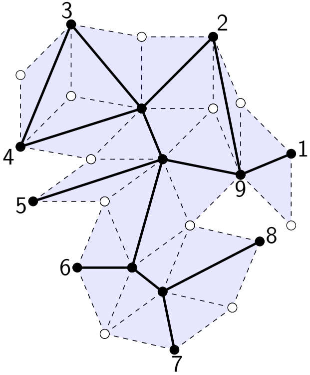
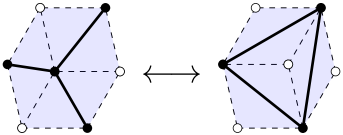
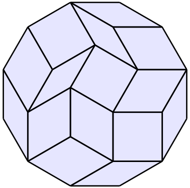
- Formula for M_R in terms of R?

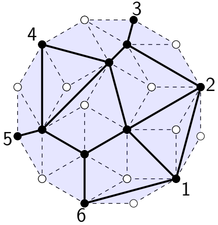




A formula for regular polygons

Let R be a regular $2N$-gon and m_{ij} be the corresponding boundary correlations.





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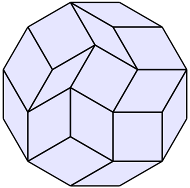

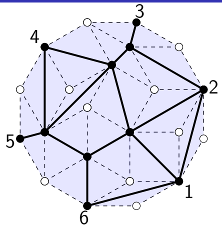

Theorem (G. (2020))

For $1 \leqslant i, j \leqslant N$ and $d := |i - j|$, we have

$$m_{ij} = \frac{2}{N}\left(\frac{1}{\sin\left((2d-1)\pi/2N\right)} - \frac{1}{\sin\left((2d-3)\pi/2N\right)} + \cdots \pm \frac{1}{\sin\left(\pi/2N\right)}\right) \mp 1.$$

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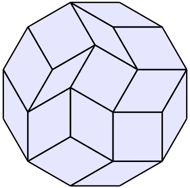

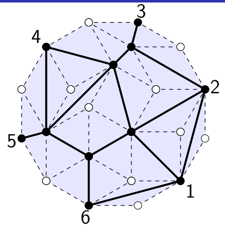

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For $1 \ll d \ll N$, this gives the Leibniz formula for π:

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots .$$

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Corollary (G. (2020))

When regular polygons approach the circle, the boundary correlations tend to the limit predicted by conformal field theory.

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Corollary (G. (2020))

When regular polygons approach the circle, the boundary correlations tend to the limit predicted by conformal field theory.

[CS12] Dmitry Chelkak and Stanislav Smirnov. Universality in the 2D Ising model and conformal invariance of fermionic observables. *Invent. Math.*, 189(3):515–580, 2012.

[Hon10] Clement Hongler. *Conformal invariance of Ising model correlations*. PhD thesis, 06/28 2010.

Theorem (G. (2020))

If R_N is a regular $2N$-gon then for $1 \leqslant i, j \leqslant N$ and $d := |i-j|$, we have

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Theorem (G. (2021))

If R is a regular $2N$-gon then for $1 \leqslant i, j \leqslant N$ and $d := |i-j|$, we have

$$\Lambda_{ij} = \frac{\sin(\pi/N)}{N \cdot \sin((2d-1)\pi/2N) \cdot \sin((2d+1)\pi/2N)}.$$

The totally nonnegative Grassmannian

$$\mathrm{Gr}(k,n) := \{\mathrm{RowSpan}(M) \mid M \text{ is a } k \times n \text{ matrix of rank } k\}.$$

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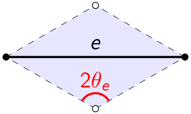
Critical Z-invariant Ising model

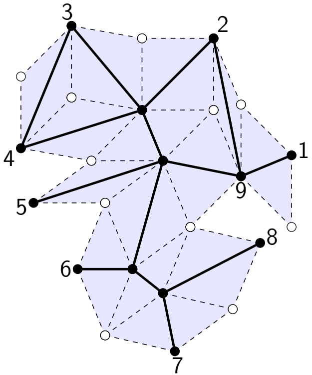
[Bax86] R. J. Baxter. Free-fermion, checkerboard and **Z**-invariant lattice models in statistical mechanics. *Proc. Roy. Soc. London Ser. A*, 404(1826):1–33, 1986.

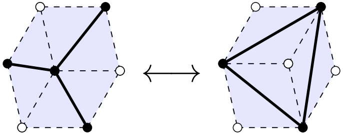
- Choose a rhombus tiling of a polygonal region R.
- G consists of diagonals connecting black vertices.
- Edge weights:



$$\longrightarrow \quad x_e = \tan(\theta_e/2)$$

- Z-invariance: these edge weights are invariant under flips (star-triangle moves).
- Conclusion: boundary correlation matrix M_R depends only on the shape of the region R.
- Formula for M_R in terms of R?





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An explicit Grassmannian formula for $\phi^{\text{Ising}}(M_R)$ in terms of the shape of R.

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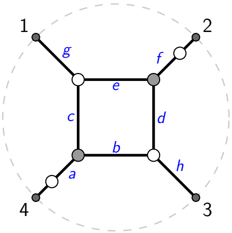
[Kar19] Steven N. Karp. Moment curves and cyclic symmetry for positive Grassmannians. *Bull. Lond. Math. Soc.*, 51(5):900–916, 2019.

[GKL17] Pavel Galashin, Steven N. Karp, and Thomas Lam. The totally nonnegative Grassmannian is a ball. *Adv. Math., to appear.* arXiv:1707.02010

Dimer model



[Pos06] Alexander Postnikov. Total positivity, Grassmannians, and networks. Preprint, `arXiv:math/0609764`, 2006.

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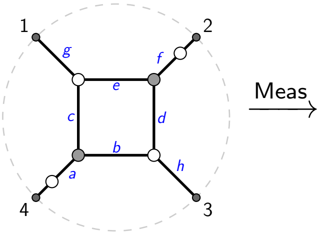

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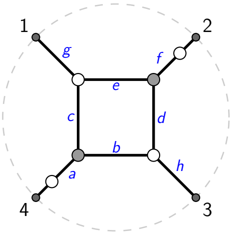

$\xrightarrow{\text{Meas}}$

$$\Delta_{12} = adg \quad \Delta_{13} = afgh \quad \Delta_{14} = bfg$$
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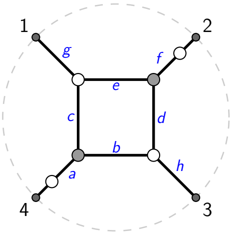

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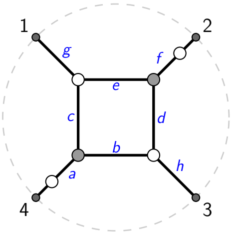

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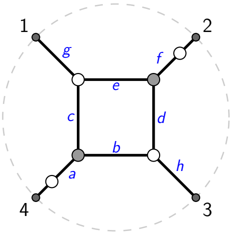

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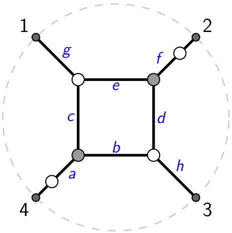

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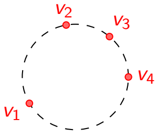

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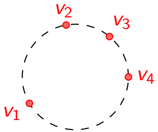

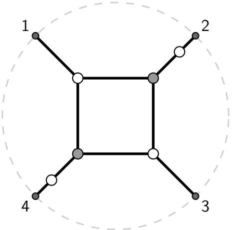

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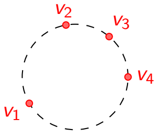

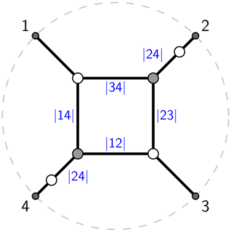

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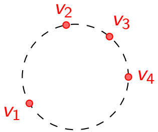

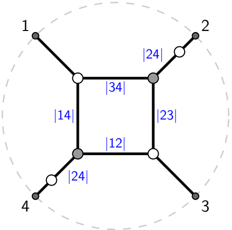

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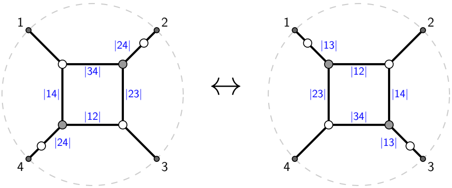

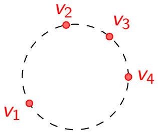

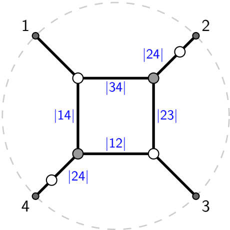

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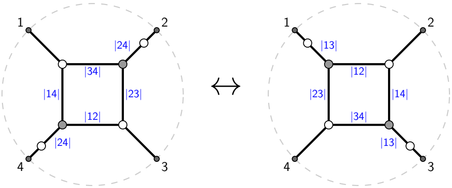
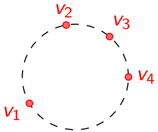
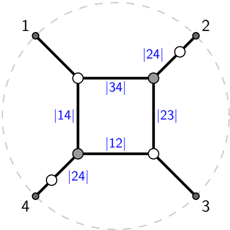


Main Result (G. (2021))

An explicit Gr(k, n) *formula for the boundary measurements of the critical dimer model.*





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Critical varieties

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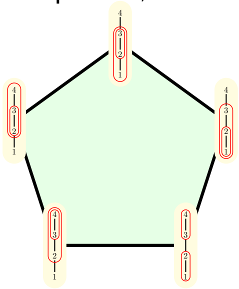
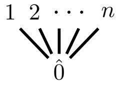


$\mathscr{A}(P)$ = associahedron



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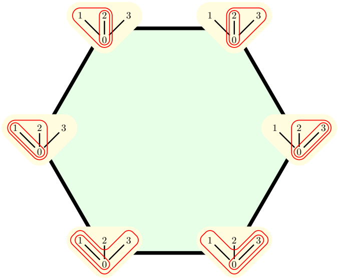
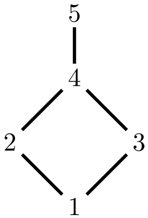


$\mathscr{A}(P)$ = permutohedron



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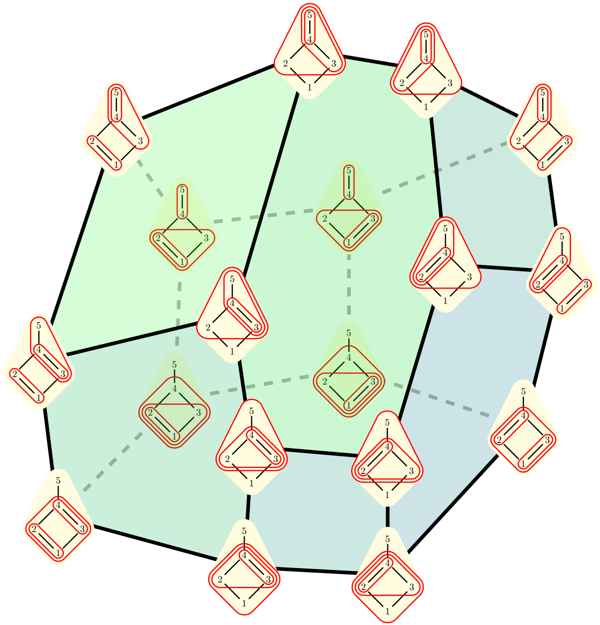
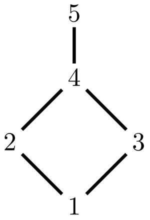


$\mathscr{A}(P)$

Thanks!



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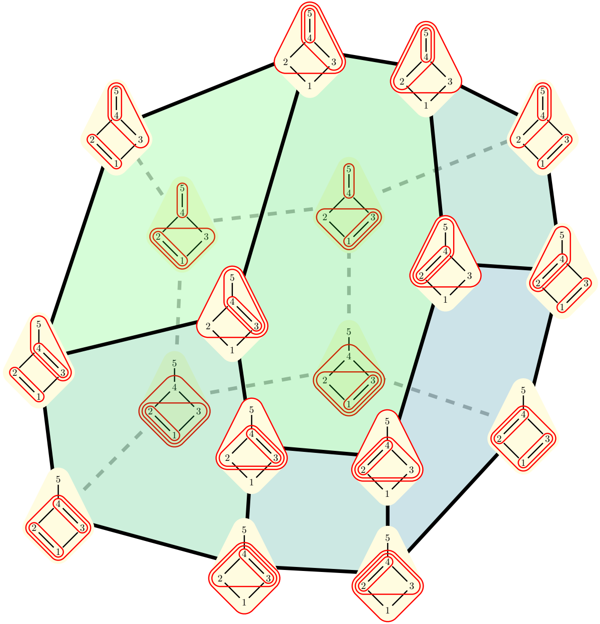

$\mathscr{A}(P)$