Refined dual stable Grothendieck polynomials and generalized Bender-Knuth involutions

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Joint work with Pavel Galashin and Darij Grinberg

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History

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- *stable Grothendieck polynomials* (Fomin-Kirillov '96): symmetric power series representatives of structure sheaves of Schubert varieties in the K-theory of the Grassmannian
- *dual stable Grothendieck polynomials* (Lam-Pylyavskyy '07): symmetric functions which are the continuous dual basis to the stable Grothendieck polynomials with respect to the Hall inner product

Reverse plane partitions

A *reverse plane partition (rpp)* is a filling of a skew diagram λ/μ with positive integers such that entries are weakly increasing along rows and columns.

		1	1	3
		1	1	
	2	2		
1	3	4		
2	3			

Irredundant content

We define the *irredundant content* of an rpp T to be the sequence $c(T) = (c_1, c_2, c_3, \dots)$ where c_i is the number of *columns* of T which contain an i.

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$$c(T) = (3, 3, 2, 1, 0, 0, \dots)$$

Dual stable Grothendieck polynomials

For each skew shape λ/μ, define

$$g_{\lambda/\mu} = \sum_{\substack{T \text{ is an rpp} \\ \text{of shape } \lambda/\mu}} x^{c(T)}$$

where $x^{(c_1,c_2,c_3,\dots)} = x_1^{c_1} x_2^{c_2} x_3^{c_3} \cdots$.

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where $x^{(c_1,c_2,c_3,\dots)} = x_1^{c_1} x_2^{c_2} x_3^{c_3} \cdots$.

The $g_{\lambda/\mu}$ are called *dual stable Grothendieck polynomials*.

Dual stable Grothendiecks are symmetric

Theorem (Lam-Pylyavskyy '07)

For every λ/μ, the power series $g_{\lambda/\mu}$ is symmetric in the x_i.

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Our result: A bijective proof of this theorem.

- Bijection is a generalization of the Bender-Knuth involutions for semistandard tableaux.

Schur functions

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$$s_{\lambda/\mu} = \sum_{\substack{T \text{ is a SSYT} \\ \text{of shape } \lambda/\mu}} x^{c(T)}.$$

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The *Bender-Knuth involutions* are a way to prove the $s_{\lambda/\mu}$ are symmetric.

Bender-Knuth involutions

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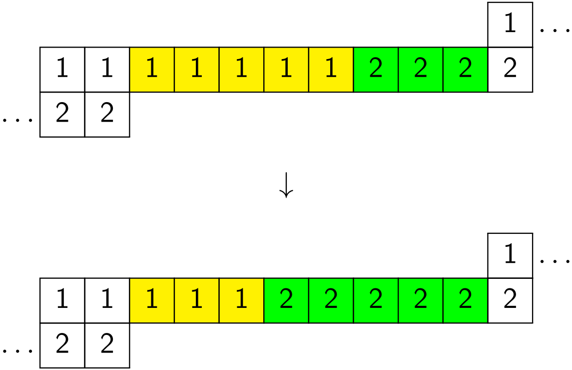
For each i, we define an involution $B_i : \mathrm{SSYT}(\lambda/\mu) \to \mathrm{SSYT}(\lambda/\mu)$ such that $c(B_i T) = s_i c(T)$, where s_i is the permutation $(i \ i+1)$.



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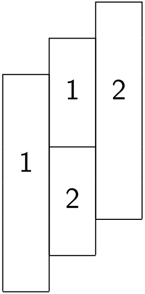
Three types of columns



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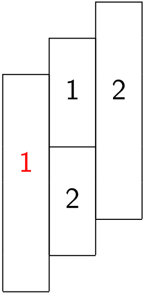

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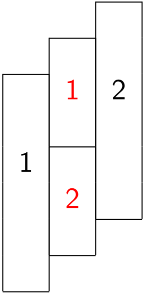

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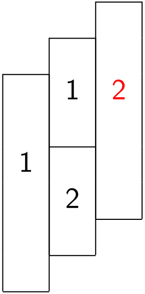

Restricting an rpp to cells with entries 1 or 2, we have three types of columns:

- *1-pure*: Contains 1's and no 2's.
- *mixed*: Contains both 1's and 2's.
- *2-pure*: Contains 2's and no 1's.

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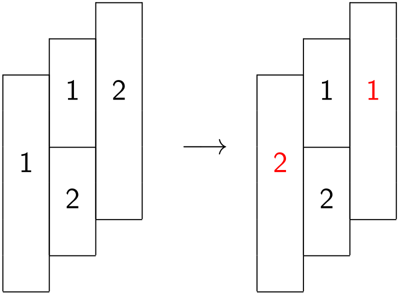

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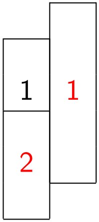
A "descent" is a pair of adjacent columns which contain a 2 immediately to the left of a 1.

Resolving descents: Example



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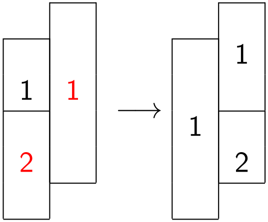

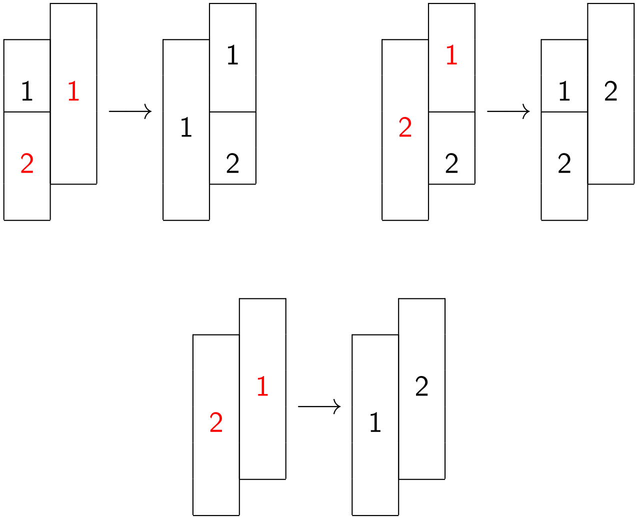
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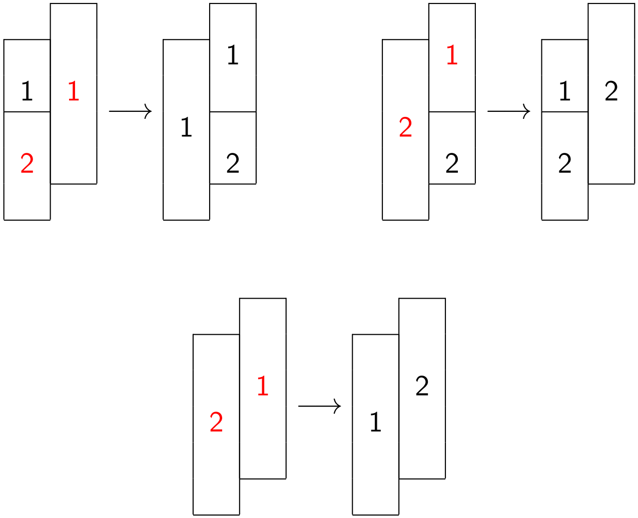

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A lemma

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Lemma

If T, T_u, and $T_v \in S$ such that $T \xrightarrow{u} T_u$ and $T \xrightarrow{v} T_v$, then there exists $T' \in S$ such that $T_u \xrightarrow{*} T'$ and $T_v \xrightarrow{*} T'$.

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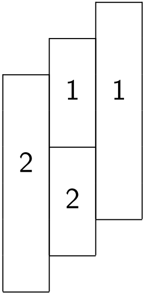
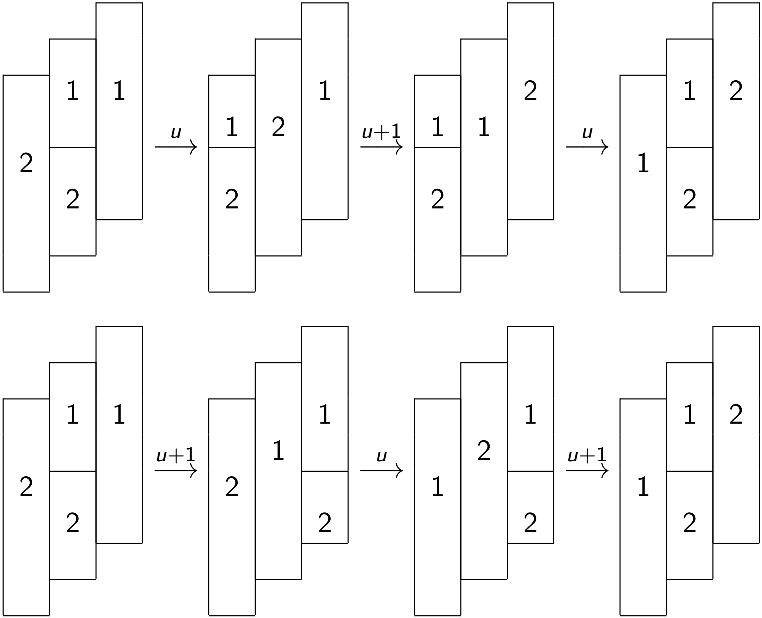
Proof: If $|u - v| \geq 2$, then the result is easy.

Assume $u = v - 1$. Columns u, $u + 1$, $u + 2$ must look like:





u
u+1
u
u+1
u
u+1

Resolving descents: End result is unique

Proposition

For each $T \in S$*, there is a unique* $T' \in \mathrm{RPP}(\lambda/\mu)$ *such that* $T \xrightarrow{*} T'$.

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By the Lemma, we must have $T'_u = T'_v$.

Since this holds for any u, v, the Proposition is proved.

Newman's Lemma

Note about the above proof: We are implicitly basing our argument on *Newman's lemma* (or the *diamond lemma*): A terminating rewriting system is confluent if it locally confluent.

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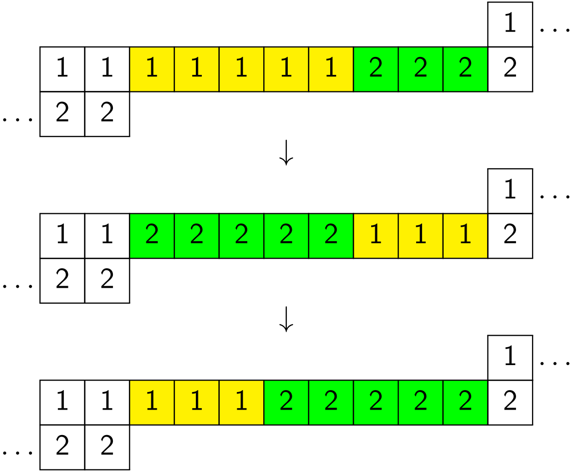
Thus, $g_{\lambda/\mu}$ is symmetric.



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Generalized Bender-Knuth involutions

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The B_i also give some additional structure to $\mathrm{RPP}(\lambda/\mu)$ beyond the above symmetry: they preserve some of the behavior between adjacent rows of an rpp.

The statistic ceq

For $T \in \mathrm{RPP}(\lambda/\mu)$, define $\mathrm{ceq}(T) = (q_1, q_2, q_3, \dots)$ where q_i is the number of vertically adjacent pairs of cells in rows i, $i+1$ of T with equal entries.

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$$\mathrm{ceq}(T) = (2, 0, 0, 1, 0, 0, \dots)$$

Refined dual stable Grothendieck polynomials

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If $t = 0$, then $\tilde{g}_{\lambda/\mu} = s_{\lambda/\mu}$.

From the previous proof, $\tilde{g}_{\lambda/\mu}$ is symmetric in x.

An example and a conjecture

Example: If λ/μ is a single column with n cells, then

$$\tilde{g}_{\lambda/\mu} = e_n(t_1, t_2, \ldots, t_{n-1}, x_1, x_2, \ldots).$$

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Conjecture (Grinberg):

$$\tilde{g}_{\lambda'/\mu'} = \det\left(e_{\lambda_i - \mu_j - i + j}(t_{\mu_j+1}, \ldots, t_{\lambda_i - 1}, x_1, x_2, \ldots)\right)_{i,j=1}^{\ell(\lambda)}$$

Thank you!

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