Plabic graphs and zonotopal tilings

Pavel Galashin

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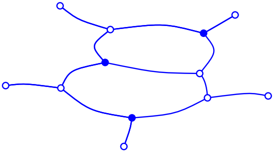
FPSAC 2018, Dartmouth College, July 19, 2018



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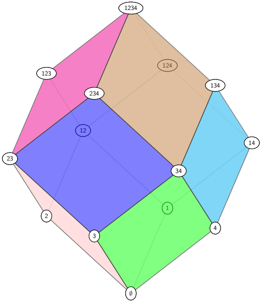

Main result

Theorem (G. (2017))

(k,n)-*plabic graphs* $\overset{\text{planar}}{\underset{\text{dual}}{\longleftrightarrow}}$ *horizontal sections at level k of fine zonotopal tilings of* $\mathcal{Z}(n,3)$

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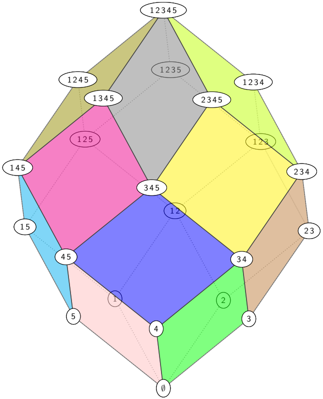

$\mathcal{Z}(n,3)$ for $n=5$

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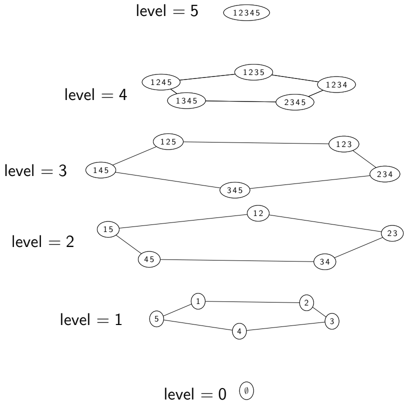

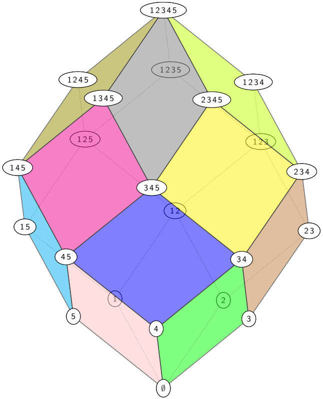

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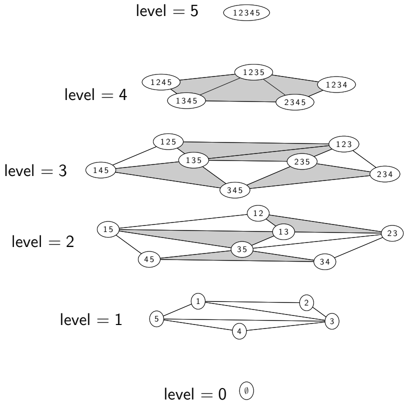

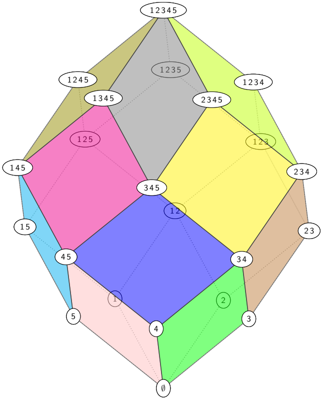

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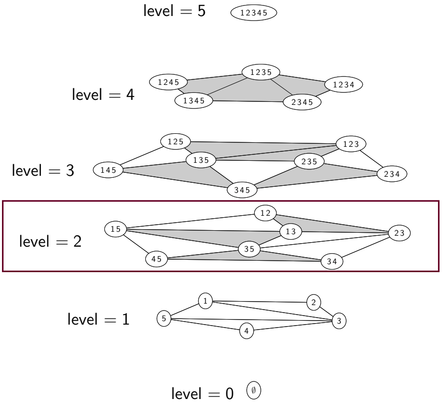

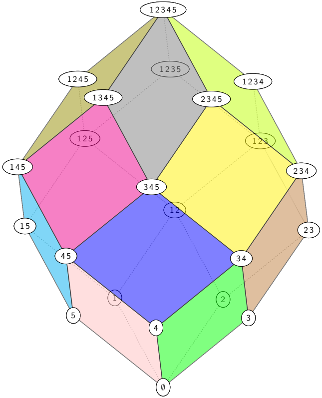

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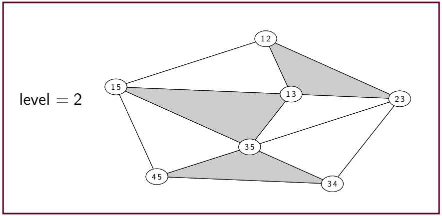

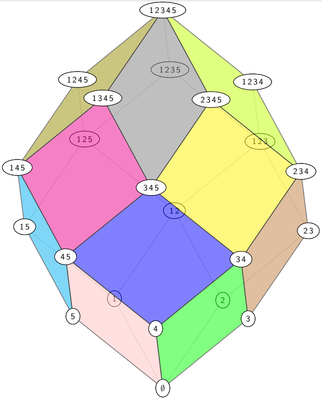

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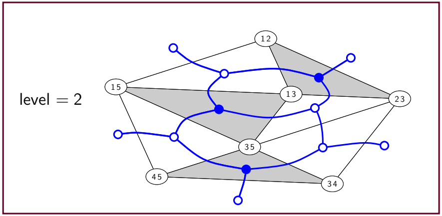

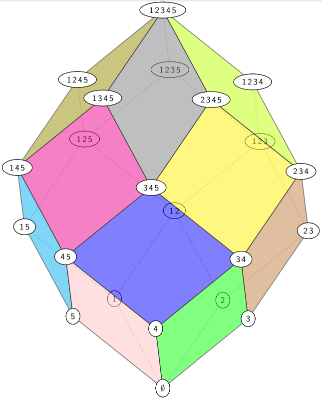

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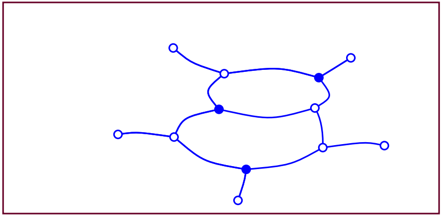
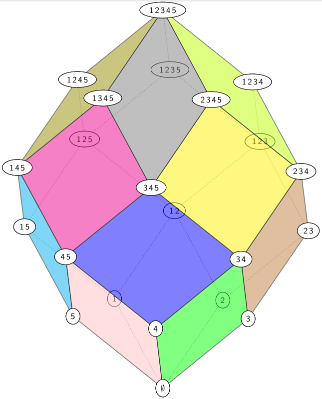
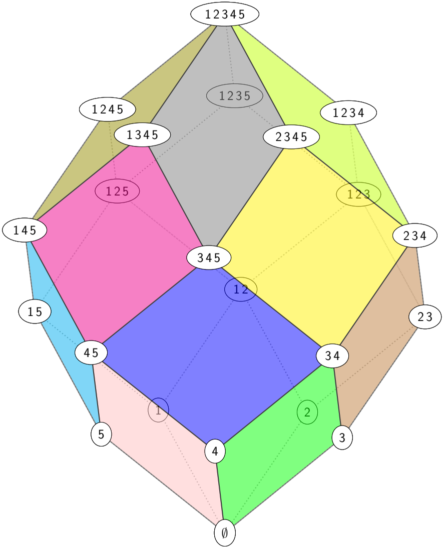


a $(2, 5)$-plabic graph



$\mathcal{Z}(n, 3)$ for $n = 5$

Part 1: Zonotopal tilings



Zonotopes

Definition (Minkowski sum)

$$A, B \subset \mathbb{R}^d, \quad A + B := \{a + b \mid a \in A, b \in B\}.$$

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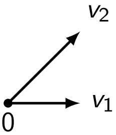
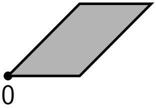
Zonotope:

$$\mathcal{Z}_{\mathbf{V}} := [0, v_1] + [0, v_2] + \cdots + [0, v_n] \subset \mathbb{R}^d.$$

Two-dimensional zonotopes

$\mathbf{V} =$ 

$\mapsto$

$\mathcal{Z}_{\mathbf{V}} =$ 

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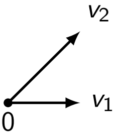

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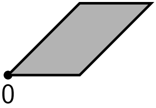

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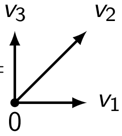

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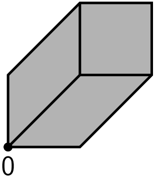

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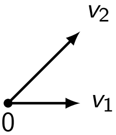

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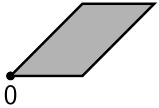

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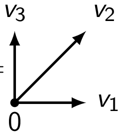

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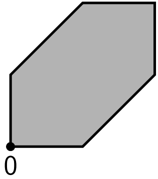

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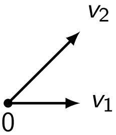
$\mathbf{V} =$ 

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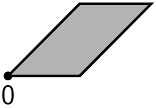
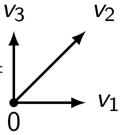
$\mathcal{Z}_{\mathbf{V}} =$ 

$\mathbf{V} =$ 

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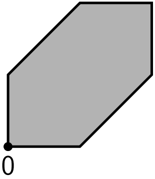
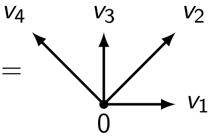
$\mathcal{Z}_{\mathbf{V}} =$ 

$\mathbf{V} =$ 

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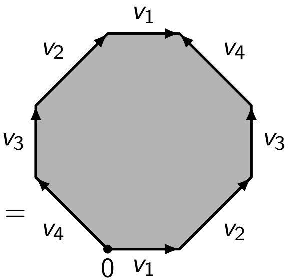
$\mathcal{Z}_{\mathbf{V}} =$ 

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Cyclic vector configuration: $\mathbf{C}(n,d) := (v_1, v_2, \ldots, v_n)$, where

$$v_i = (1, r_i, r_i^2, \ldots, r_i^{d-1}) \quad \text{for some } 0 < r_1 < r_2 < \cdots < r_n \in \mathbb{R}.$$

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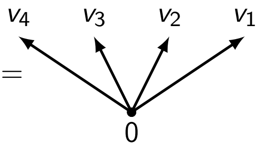
$\mathbf{C}(4,2) =$ 

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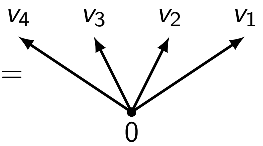

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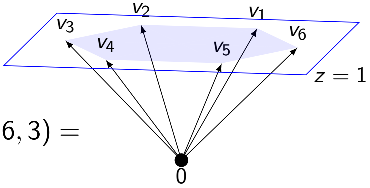

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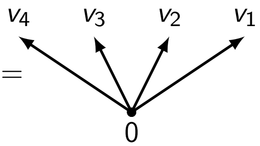
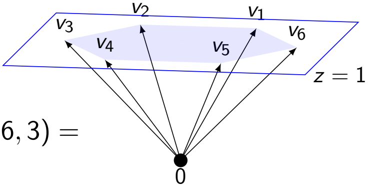
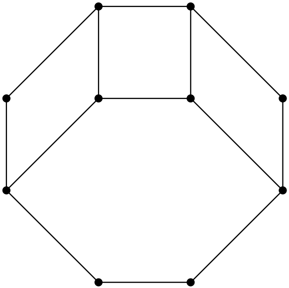
$\mathbf{C}(4,2) =$ 

$\mathbf{C}(6,3) =$ 

Zonotopal tilings

Definition

A *zonotopal tiling* of $\mathcal{Z}_{\mathbf{V}}$ is a polyhedral subdivision Δ of $\mathcal{Z}_{\mathbf{V}}$ into smaller zonotopes.



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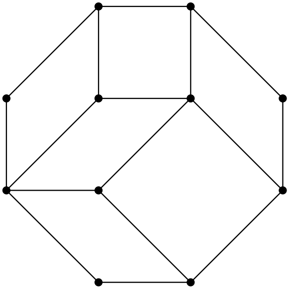
A zonotopal tiling is *fine* if all pieces are parallelotopes.



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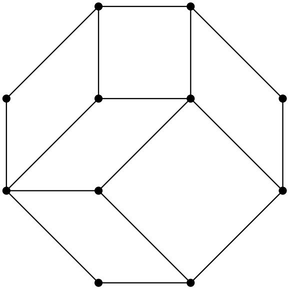

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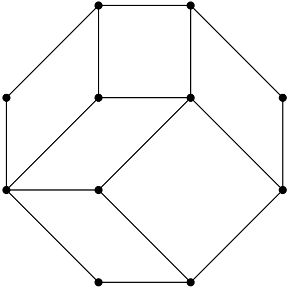

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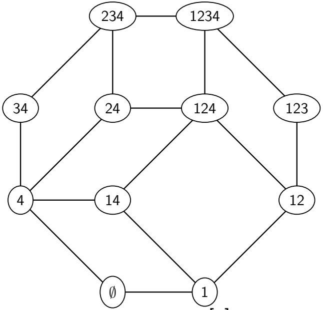

$\text{Vert}(\Delta) \subset 2^{[n]}$

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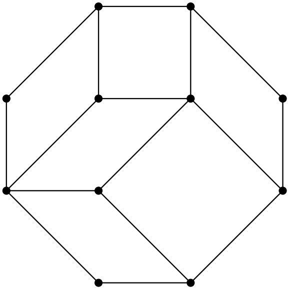

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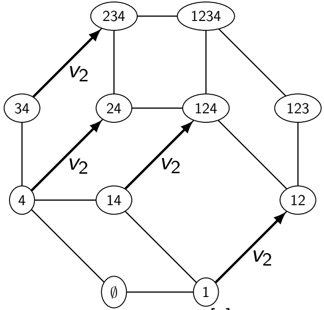

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Vertices of zonotopal tilings

Fact

Number of vertices in a fine zonotopal tiling of $\mathcal{Z}_{\mathbf{V}}$ equals the number $\mathrm{Ind}(\mathbf{V})$ *of linearly independent subsets of* $\mathbf{V}$.

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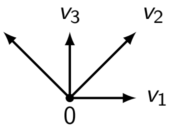

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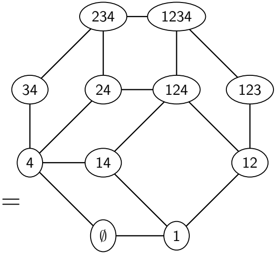

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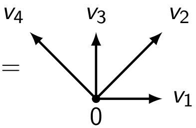
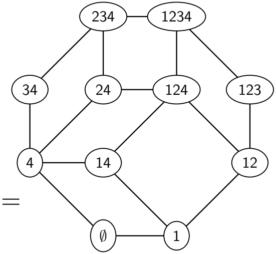
$\mathbf{V} =$ 

$\Delta =$ 

$$\mathrm{Ind}(\mathbf{V}) = \binom{4}{0} + \binom{4}{1} + \binom{4}{2} = 11, \quad |\mathrm{Vert}(\Delta)| = 11.$$

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Which collections of subsets of $[n]$ *can appear as* $\mathrm{Vert}(\Delta)$*, where* Δ *is a fine zonotopal tiling of* $\mathcal{Z}(n,2)$*?*

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$S, T \subset [n]$ are *strongly separated* if there is no $i < j < k$ such that

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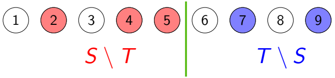

Vertices of zonotopal tilings

Question

Which collections of subsets of $[n]$ *can appear as* $\mathrm{Vert}(\Delta)$*, where* Δ *is a fine zonotopal tiling of* $\mathcal{Z}(n,2)$*?*

Definition (Leclerc–Zelevinsky (1998))

$S, T \subset [n]$ are *strongly separated* if there is no $i < j < k$ such that

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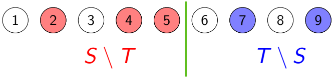
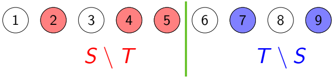


$\mathcal{D} \subset 2^{[n]}$ is a *strongly separated collection* if all $S, T \in \mathcal{D}$ are strongly separated.

Purity phenomenon

Strongly separated:



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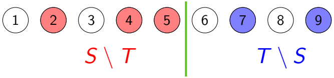

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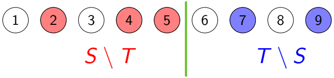

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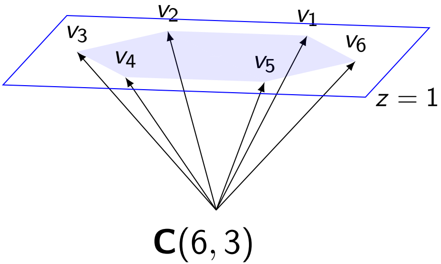
Corollary (Leclerc–Zelevinsky (1998))

Purity phenomenon: every maximal by inclusion strongly separated collection $\mathcal{D} \subset 2^{[n]}$ is also maximal by size:

$$|\mathcal{D}| = \binom{n}{0} + \binom{n}{1} + \binom{n}{2}.$$

3D zonotopes



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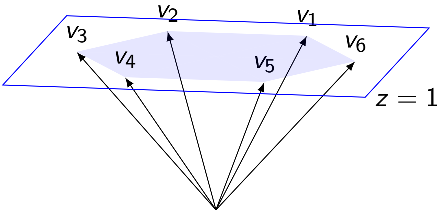
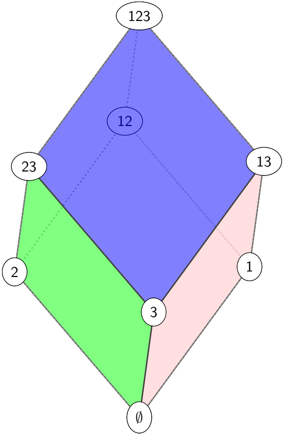
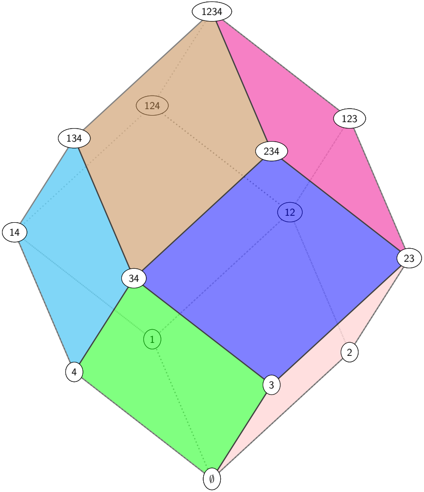


$\mathbf{C}(6,3)$



$\mathcal{Z}(3,3)$

3D zonotopes: $\mathcal{Z}(4,3)$



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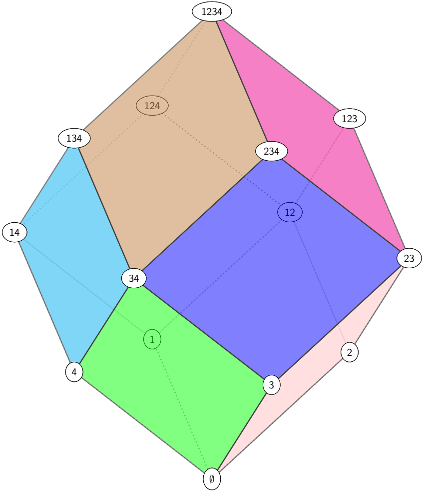

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Q: How many fine zonotopal tilings?

Chord separation

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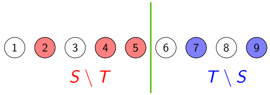

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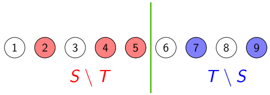

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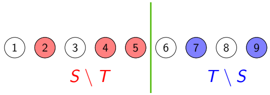

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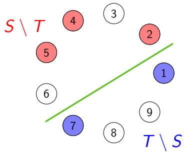

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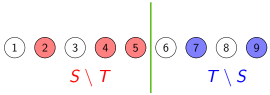

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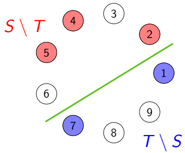

When $|S| = |T|$, both definitions are due to Leclerc–Zelevinsky.

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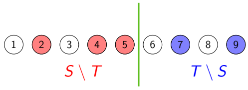

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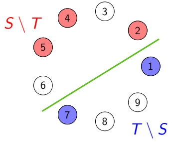

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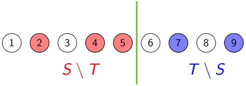

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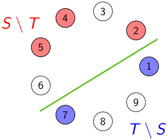

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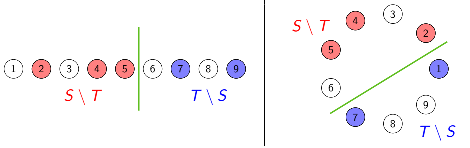

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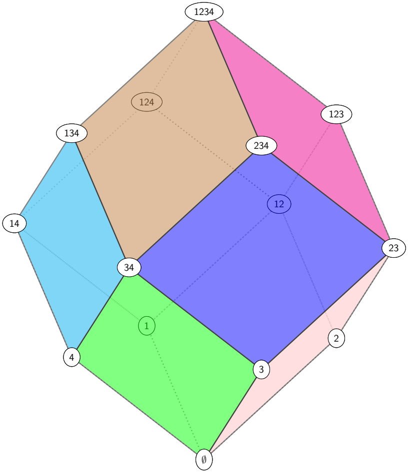

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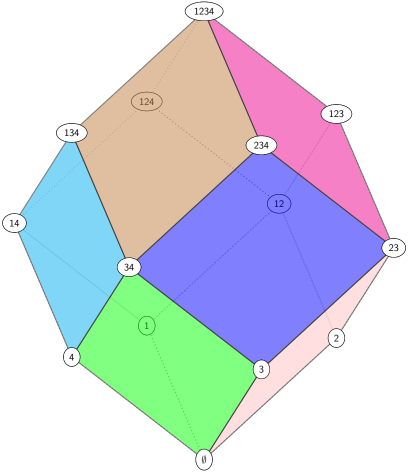
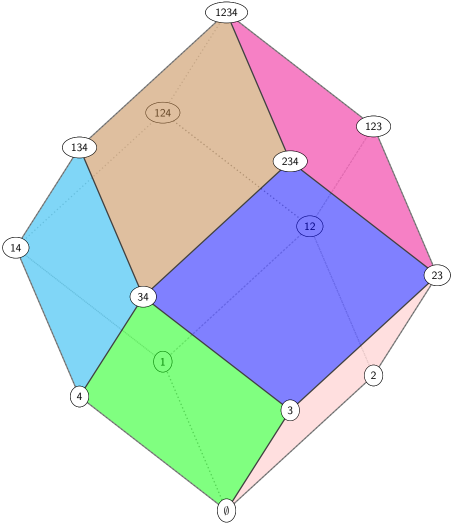
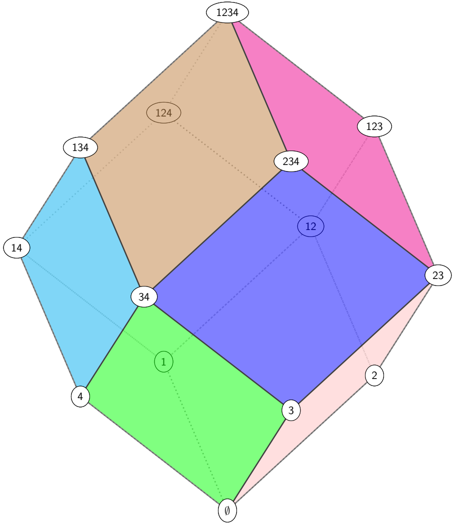


Q: How many fine zonotopal tilings?

A: Two.

Fine zonotopal tilings of $\mathcal{Z}(4,3)$





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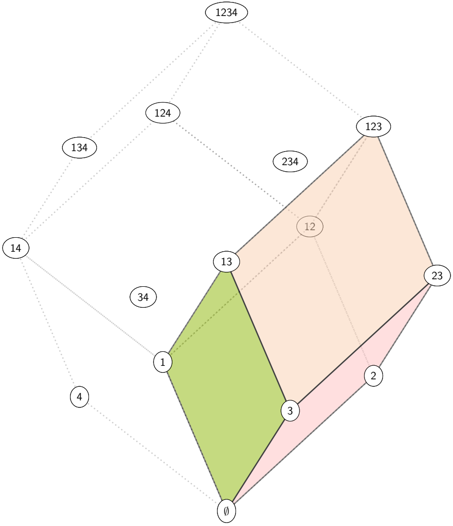

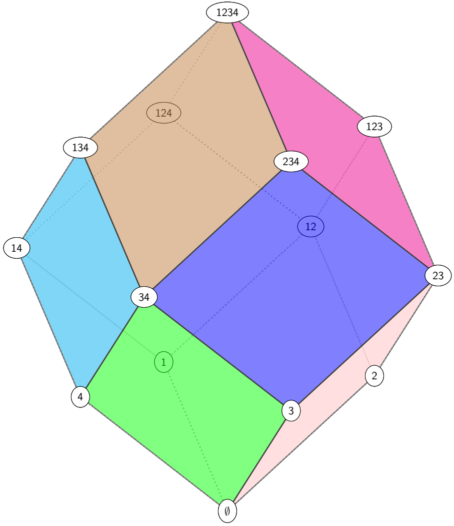

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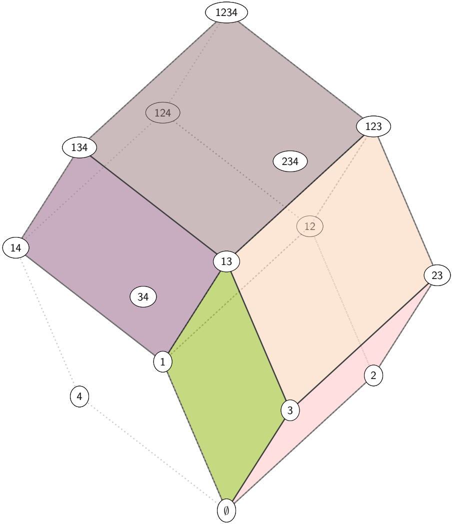

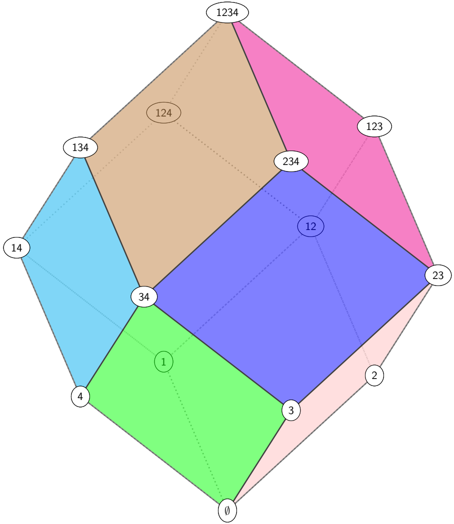

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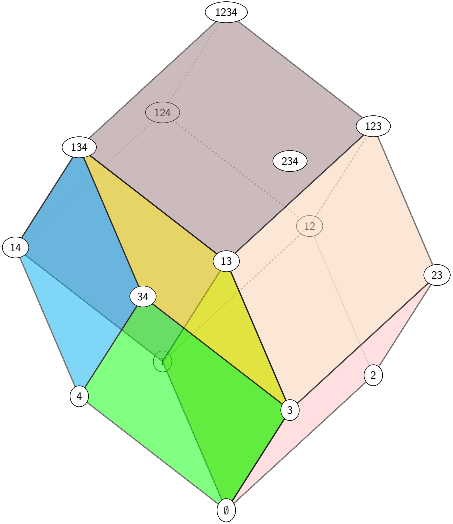

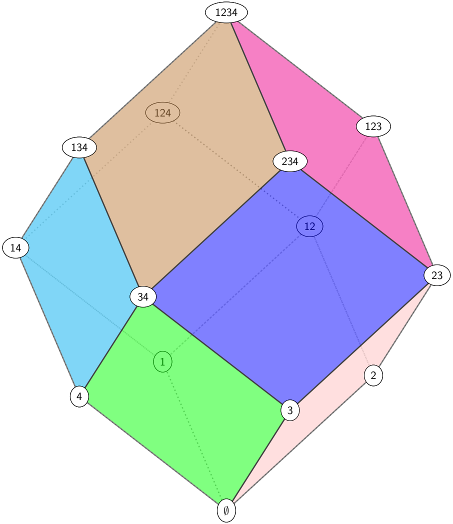

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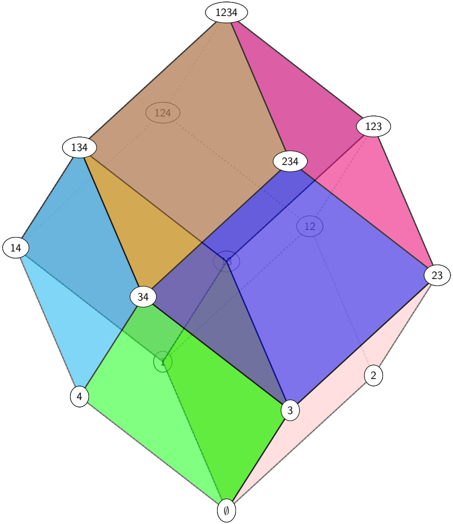

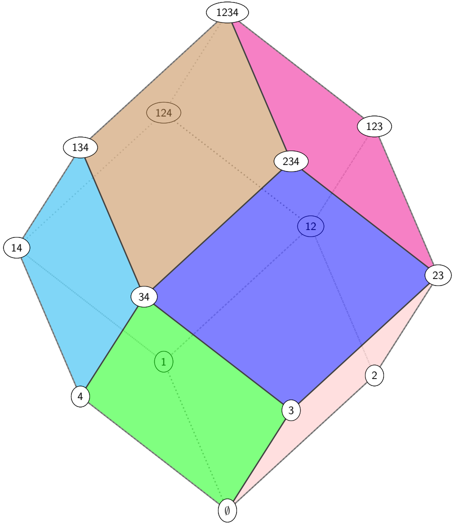

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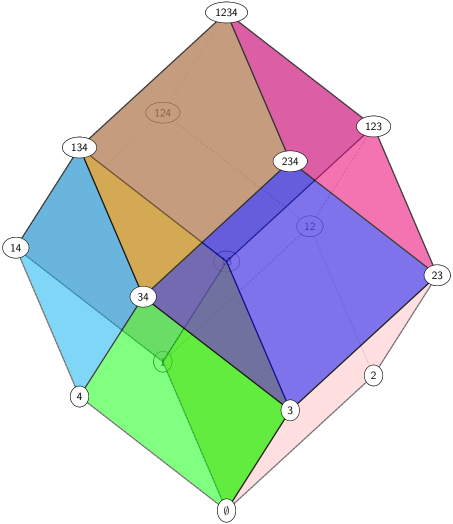

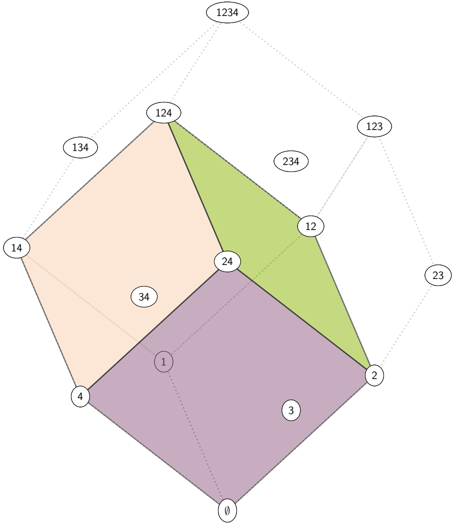

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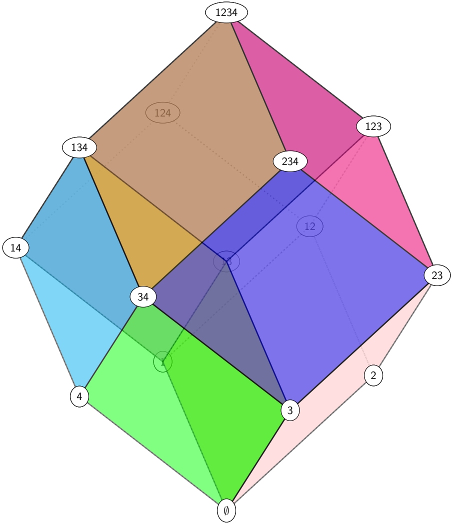

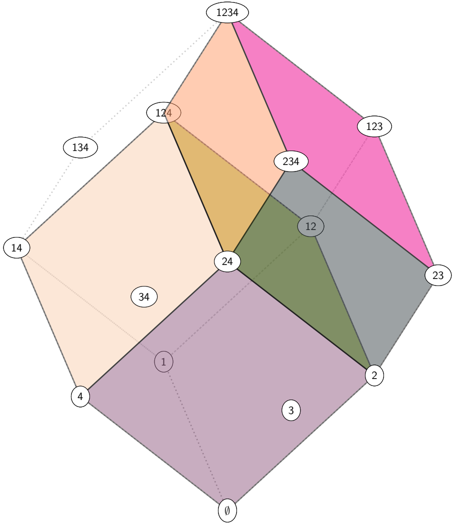

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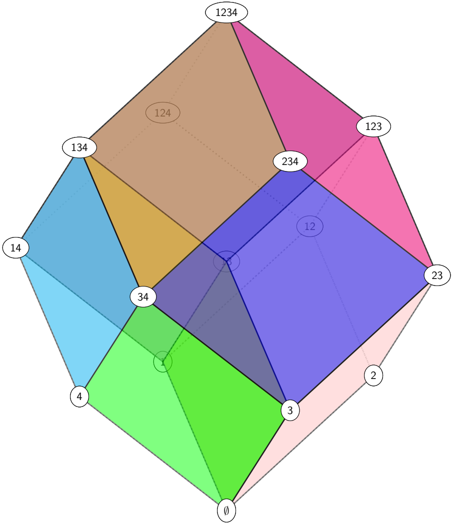

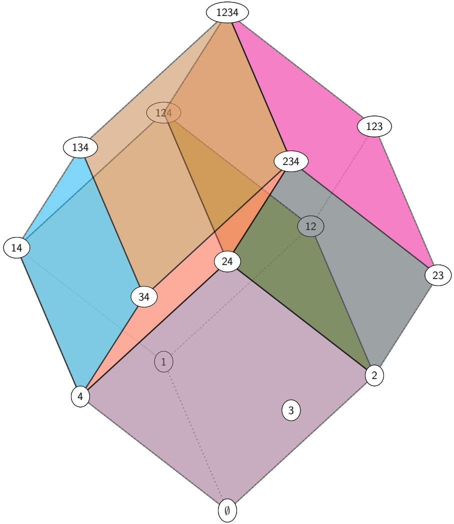

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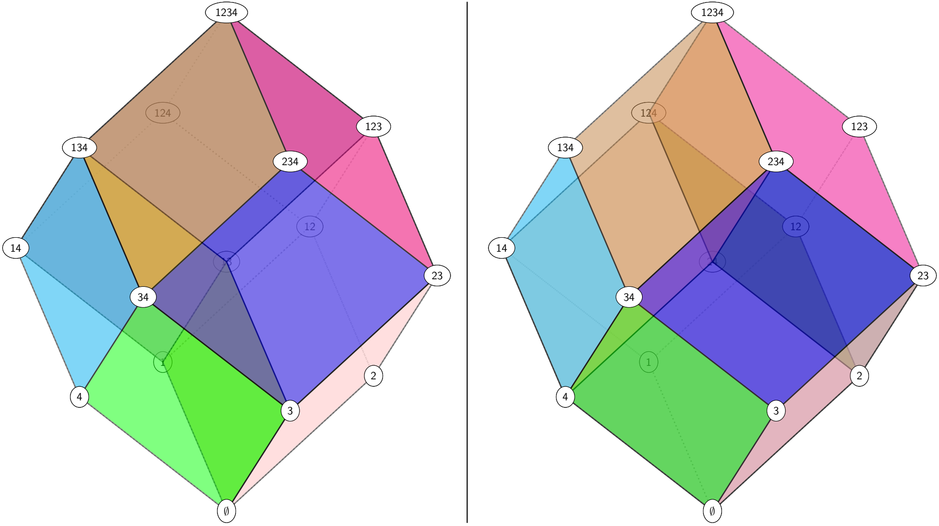
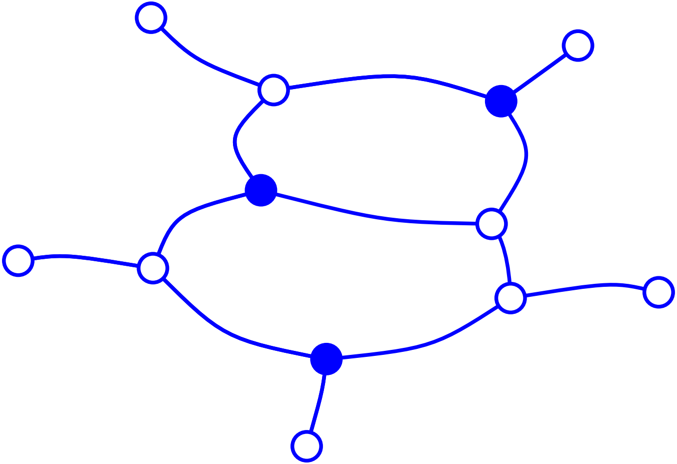


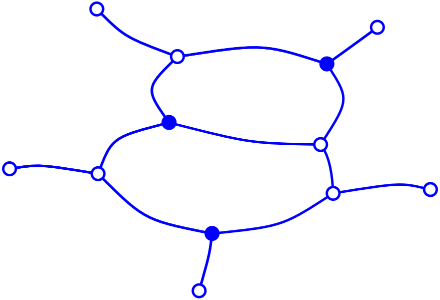
Part 2: Plabic graphs



Plabic graphs and strands

Definition (Postnikov (2007))

A *plabic graph* is a planar graph embedded in a disk, with n boundary vertices of degree 1, and the remaining vertices all trivalent and colored black and white.



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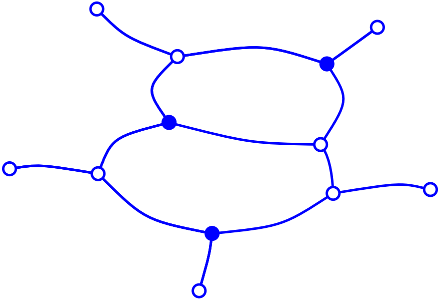

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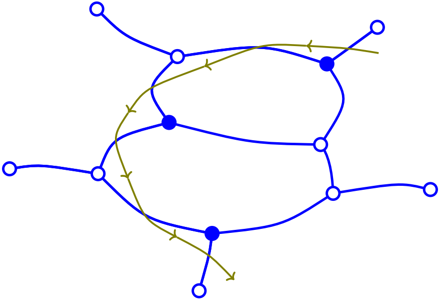

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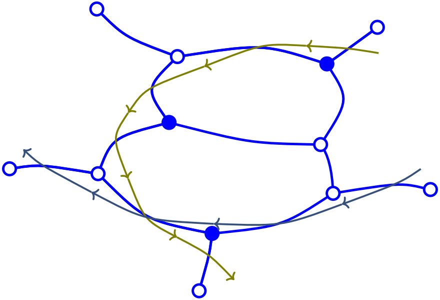

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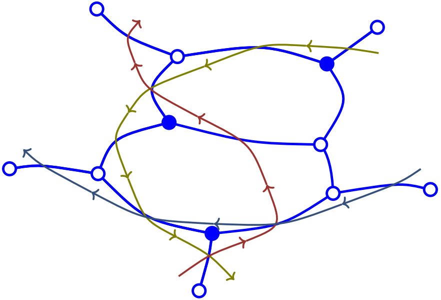

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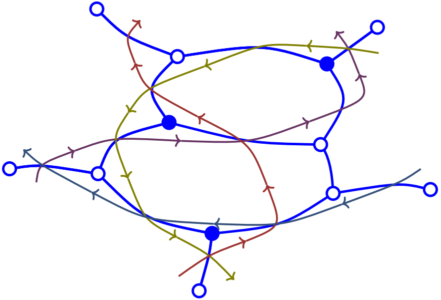

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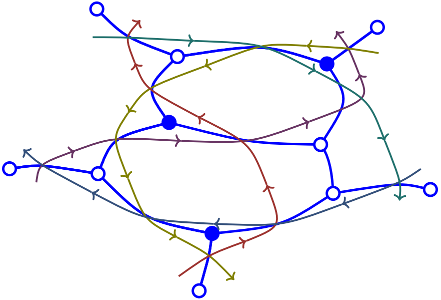

(k, n)-plabic graphs

Definition (Postnikov (2007))

A plabic graph is *reduced* if it contains:

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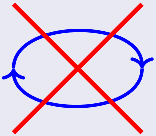
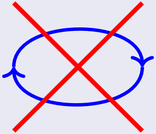


No closed strands

(k, n)-plabic graphs

Definition (Postnikov (2007))

A plabic graph is *reduced* if it contains:



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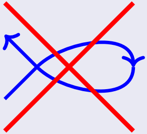

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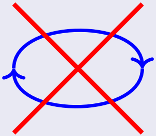

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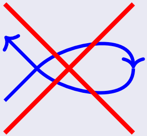

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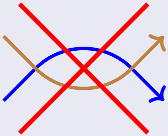

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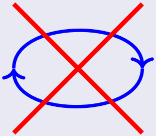

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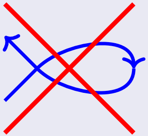

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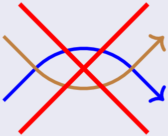

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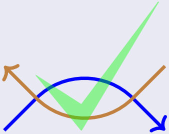

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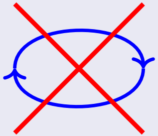

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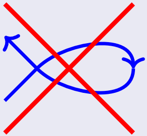

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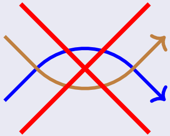

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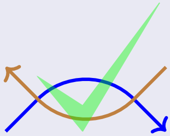

"Good double crossings" are OK!

A *(k, n)-plabic graph* is a reduced plabic graph such that:

- the strand that starts at i ends at $i + k$ modulo n for all i.

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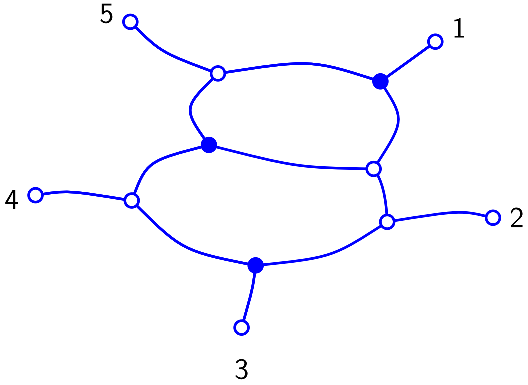

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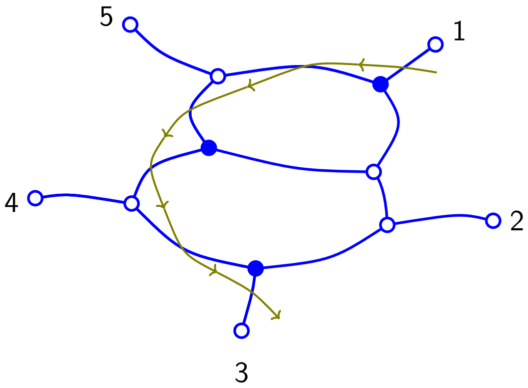

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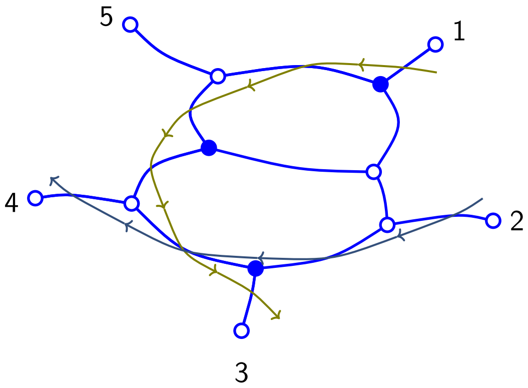

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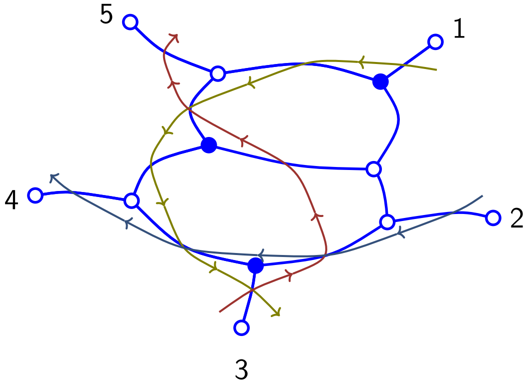

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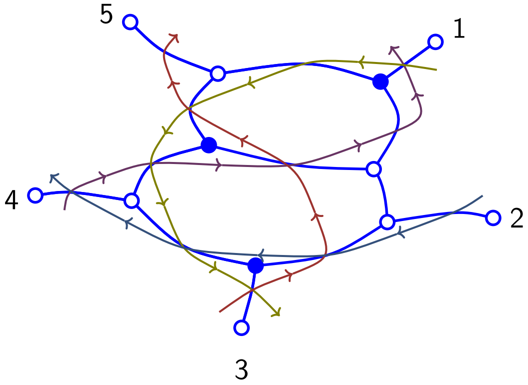

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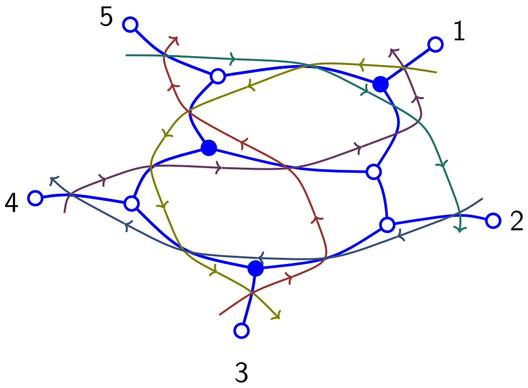
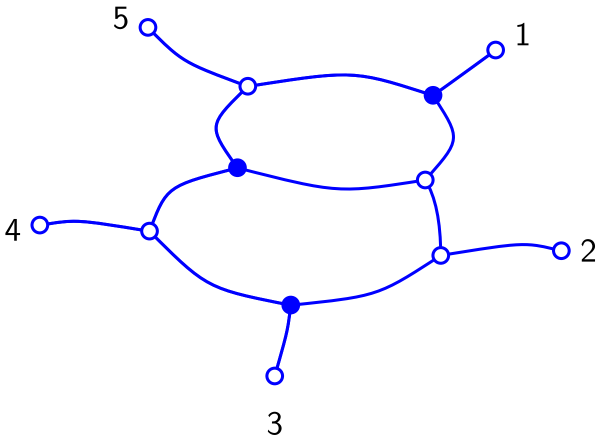


Face labels

Postnikov (2007): each (k, n)-plabic graph has $k(n-k)+1$ faces.



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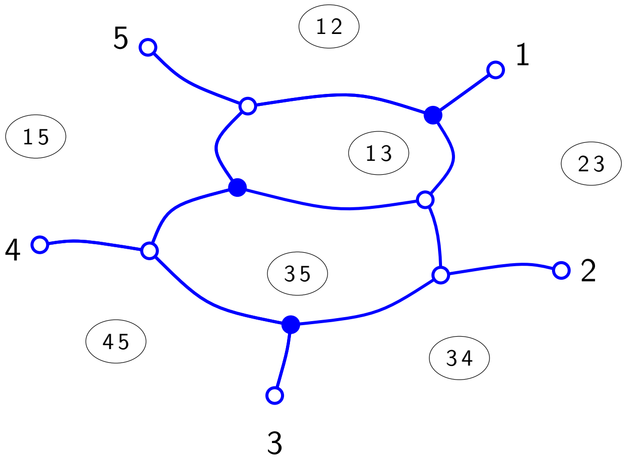

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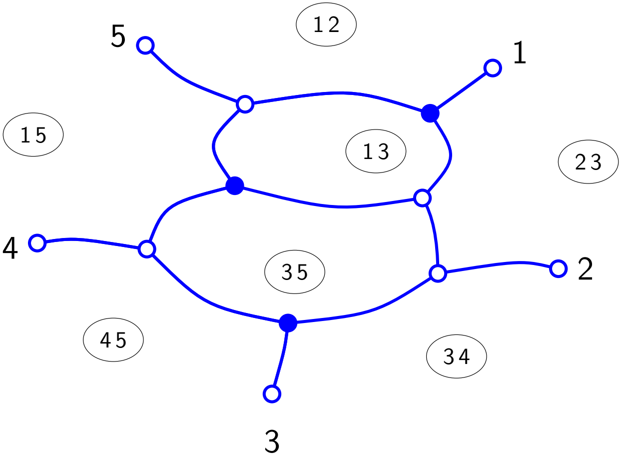

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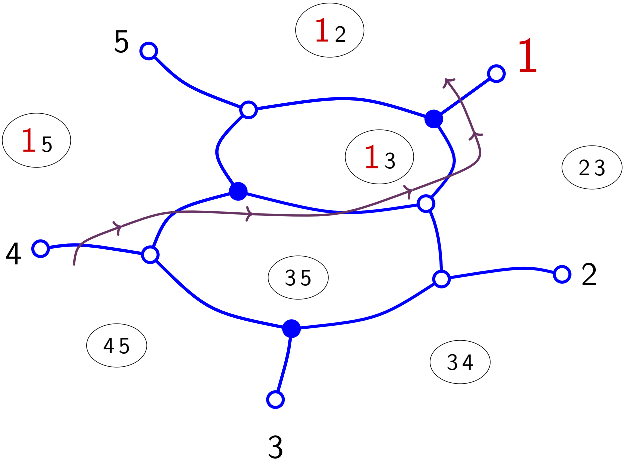

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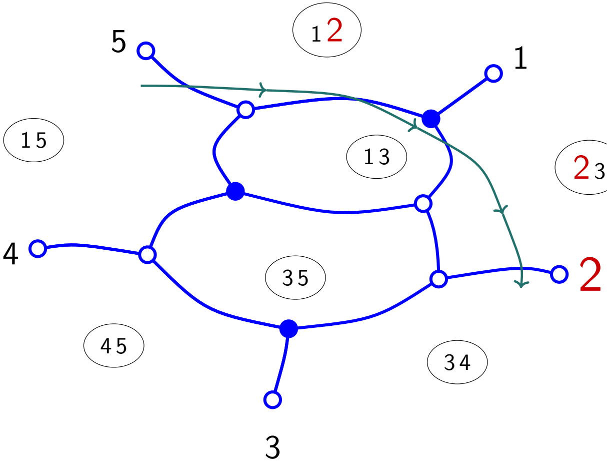

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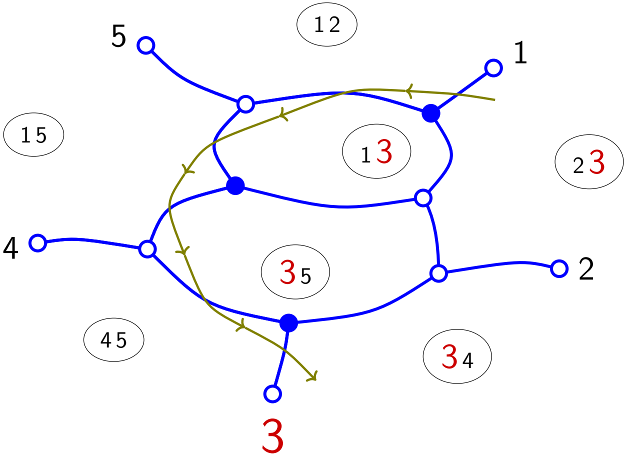

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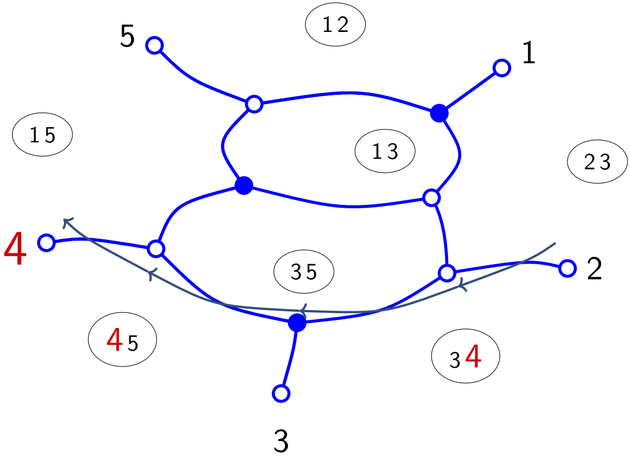

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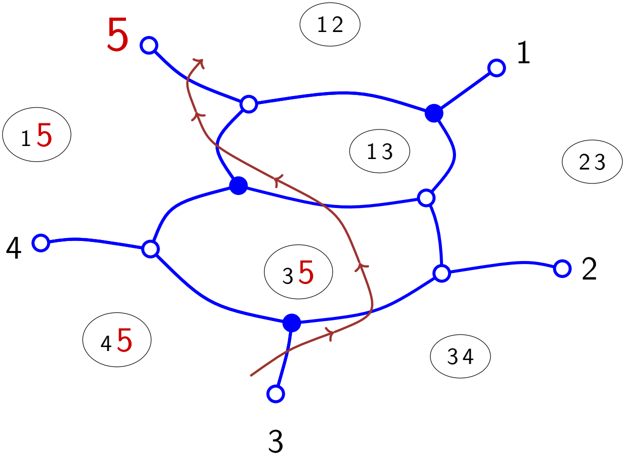

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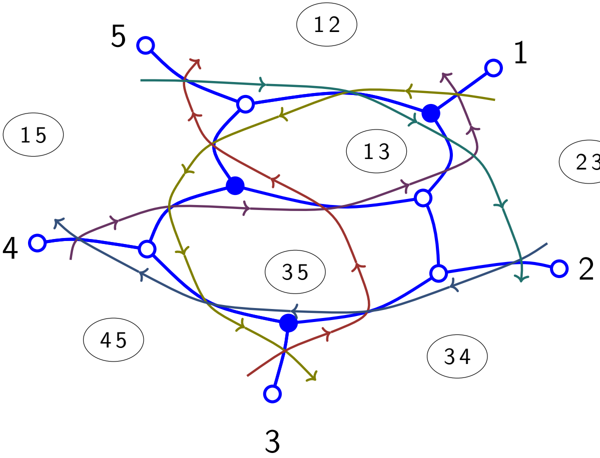

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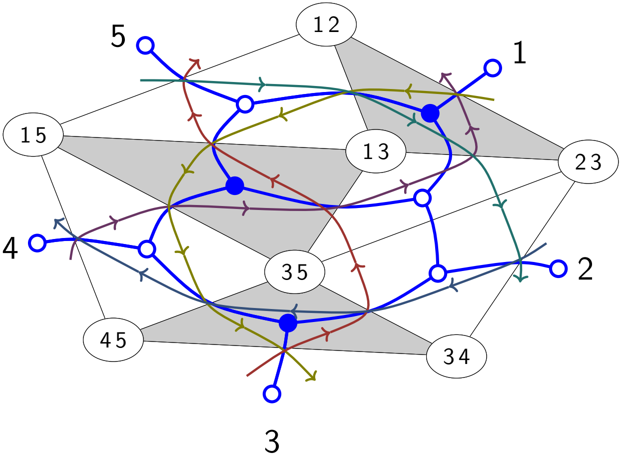

Plabic graphs and chord separation

Conjecture (Leclecrc–Zelevinsky (1998), Scott (2005))

Every maximal by inclusion chord separated collection $\mathcal{D} \subset \binom{[n]}{k}$ *has size*

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Theorem (Oh–Postnikov–Speyer (2011))

The map $G \mapsto \text{Faces}(G)$ *is a bijection** *between:*

- (k,n)*-plabic graphs, and*
- *maximal by inclusion chord separated collections* $\mathcal{D} \subset \binom{[n]}{k}$.

Contradction?

Corollary (Oh–Postnikov–Speyer (2011))

Every maximal by inclusion chord separated collection $\mathcal{D} \subset \binom{[n]}{k}$ *has size*

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Contradiction?

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Theorem (G. (2017))

The map $\Delta \mapsto \operatorname{Vert}(\Delta)$ *is a bijection between:*

- *fine zonotopal tilings* Δ *of* $\mathcal{Z}(n,3)$, *and*
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$$\mathrm{Ind}(\mathbf{C}(n,3)) = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \binom{n}{3}.$$

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Luckily for us,

$$\binom{n}{0}+\binom{n}{1}+\binom{n}{2}+\binom{n}{3}=\sum_{k=0}^{n}(k(n-k)+1).$$

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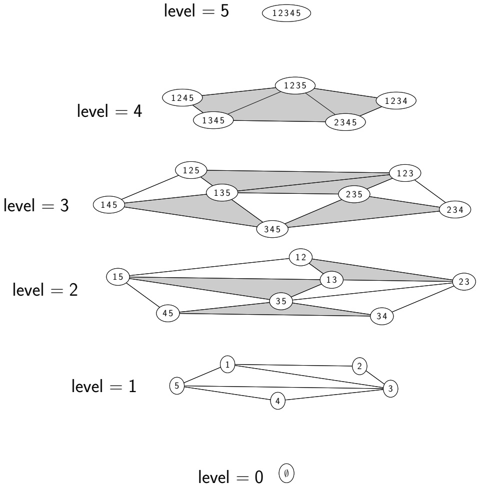
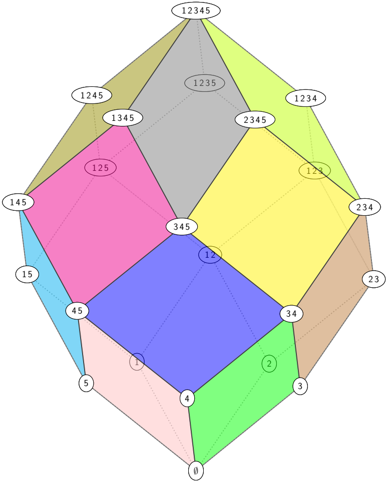
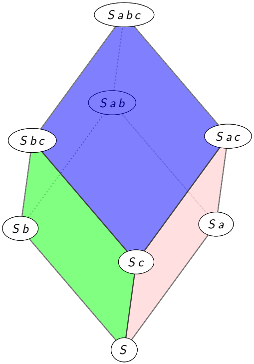
Part 3: Putting it all together





Sections of tiles



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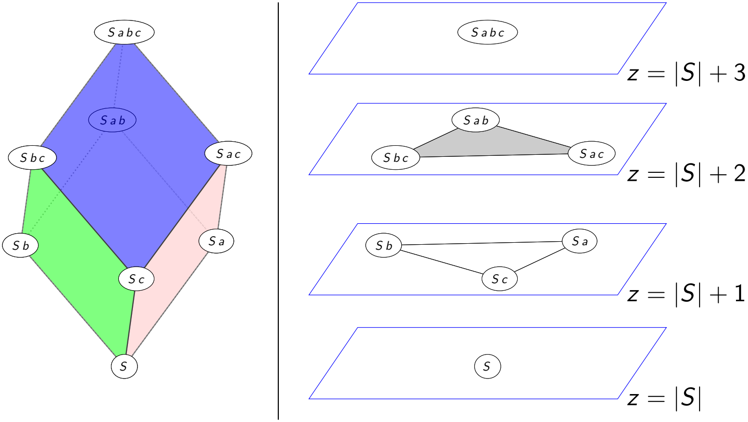
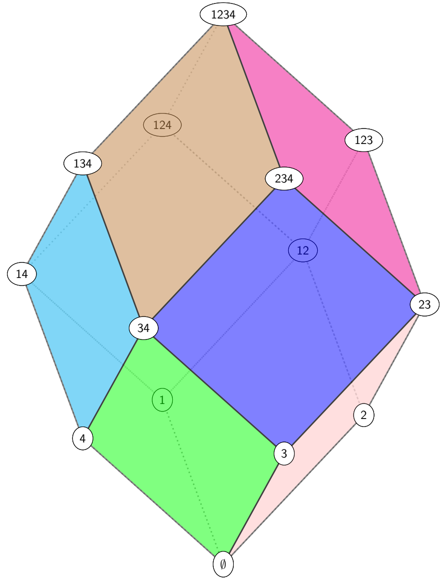


Example: $n = 4$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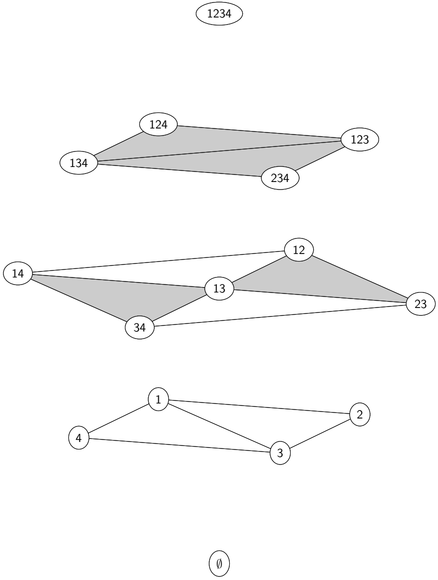
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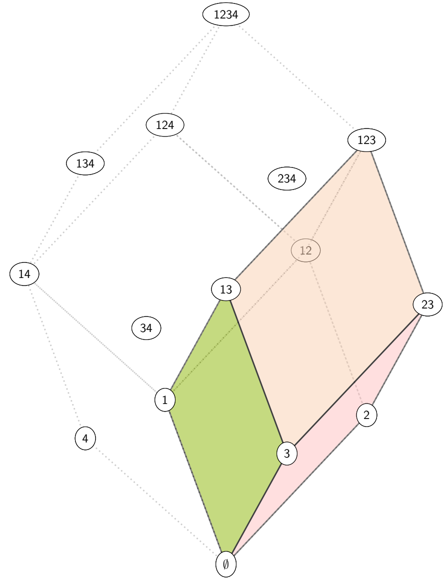

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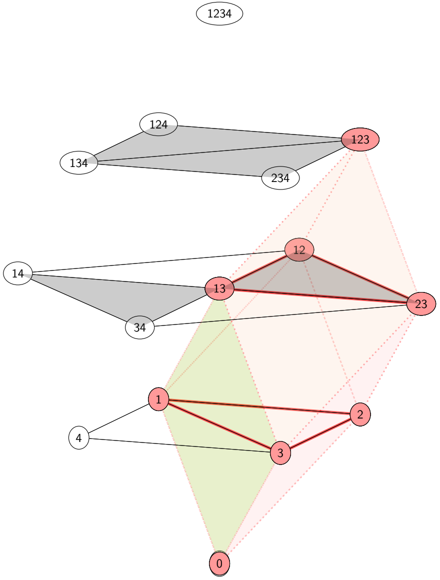

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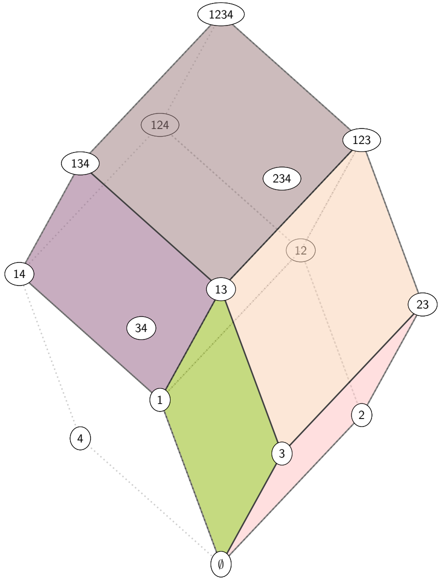

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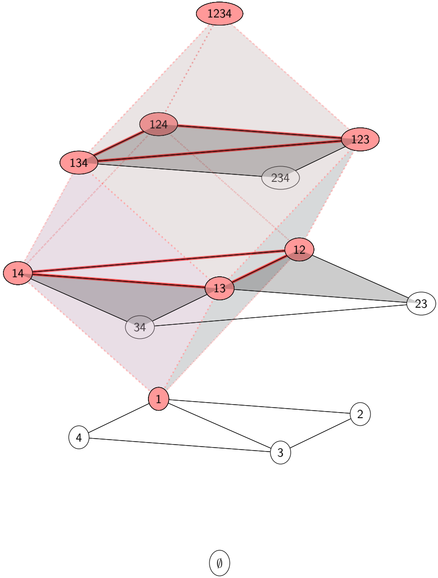

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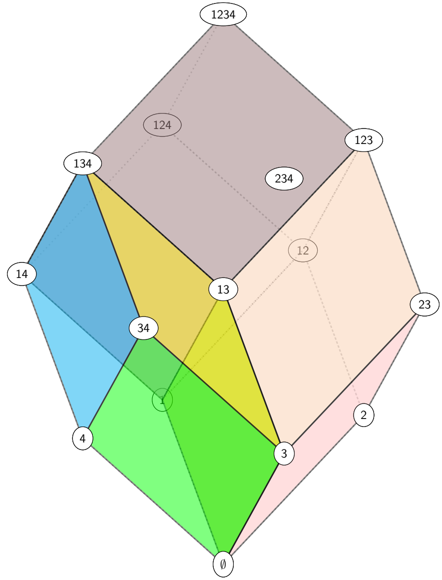
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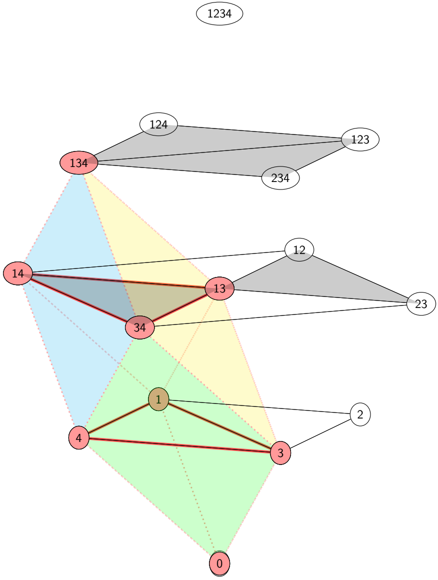
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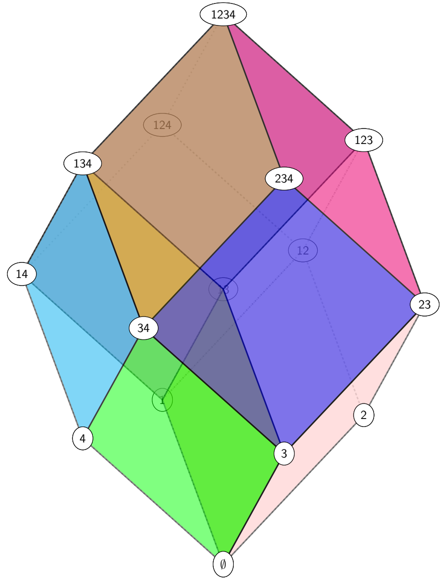
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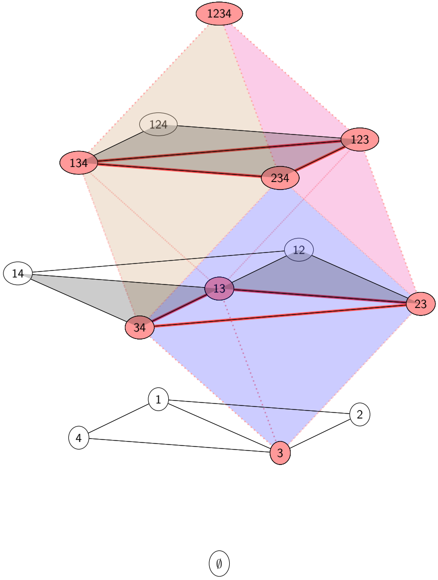
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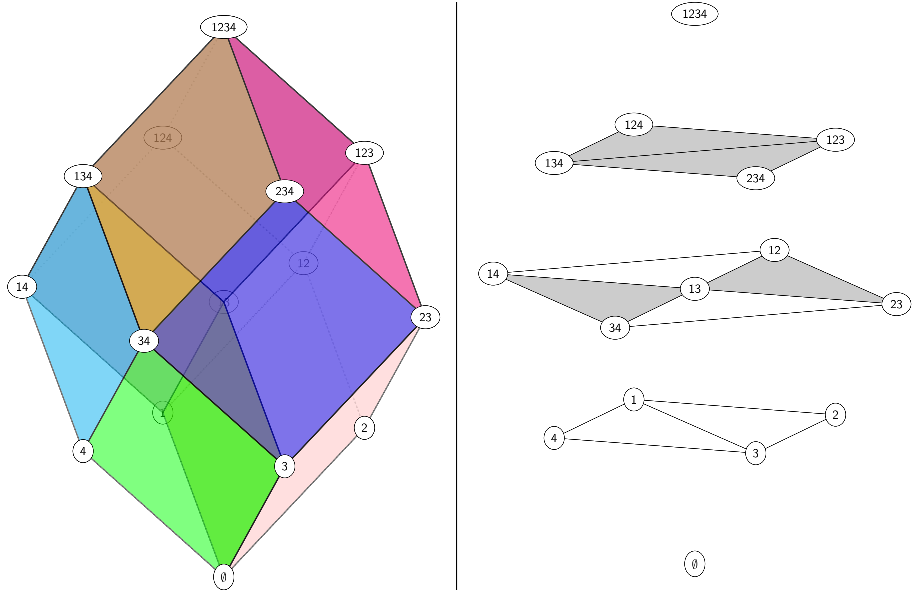
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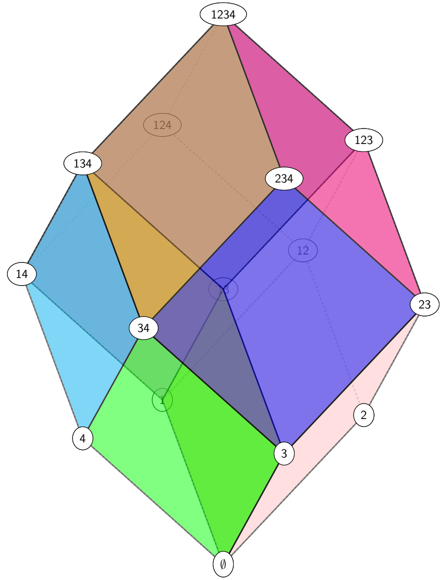

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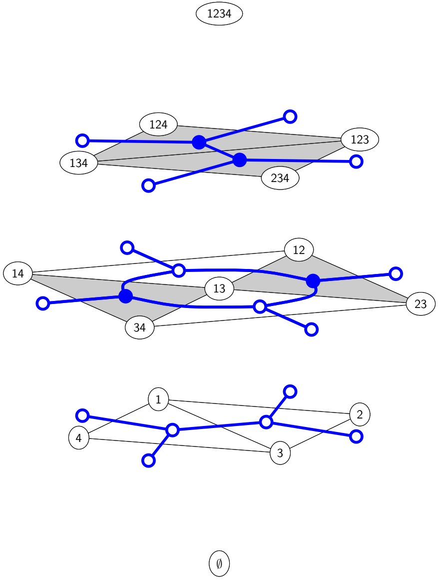

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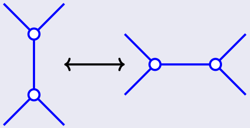
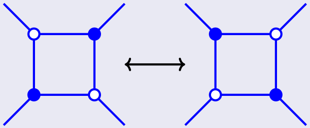
Moves and flips

Theorem (Postnikov (2007))

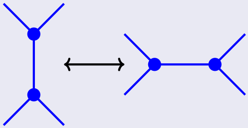
Any two (k, n)-plabic graphs are connected by a sequence of moves:



(M1)



(M2)

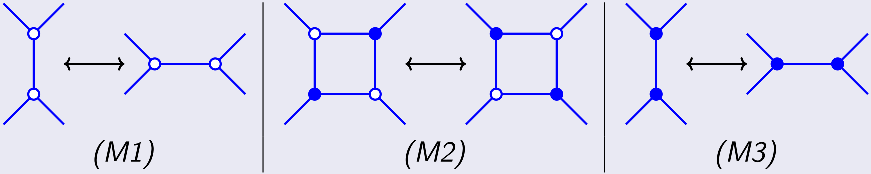


(M3)

Moves and flips

Theorem (Postnikov (2007))

Any two (k, n)-plabic graphs are connected by a sequence of moves:



(M1) (M2) (M3)

A *flip* of a fine zonotopal tiling of $\mathcal{Z}(n, 3)$ consists of replacing one tiling of $\mathcal{Z}(4, 3)$ with another.

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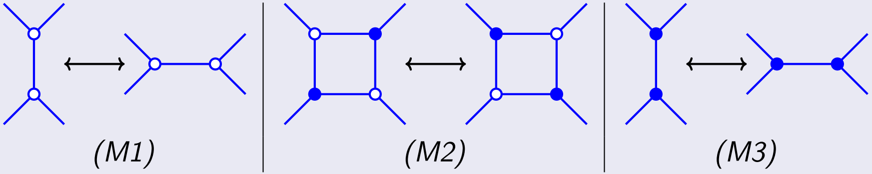

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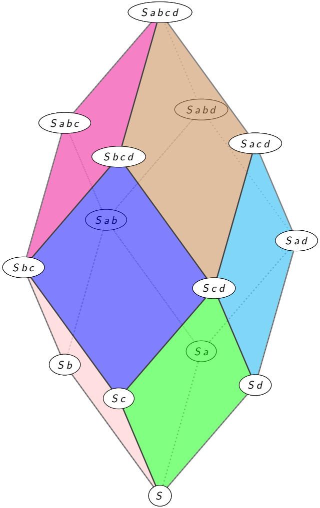
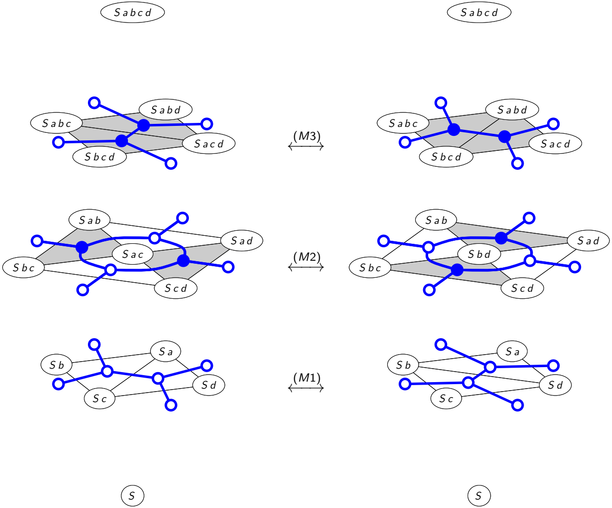
Theorem (Ziegler (1993))

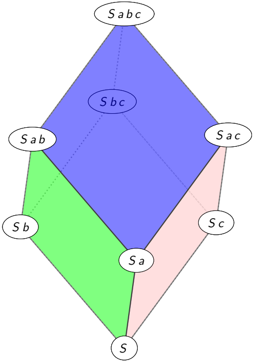
Any two fine zonotopal tilings of $\mathcal{Z}(n, 3)$ are connected by a sequence of flips.

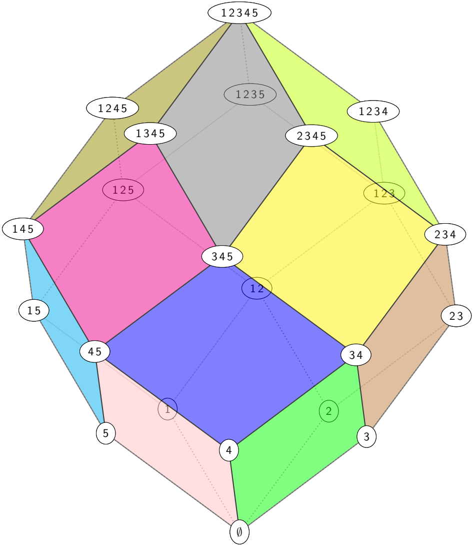
Moves = sections of flips



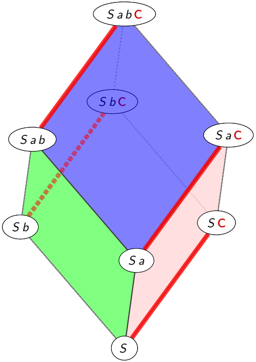


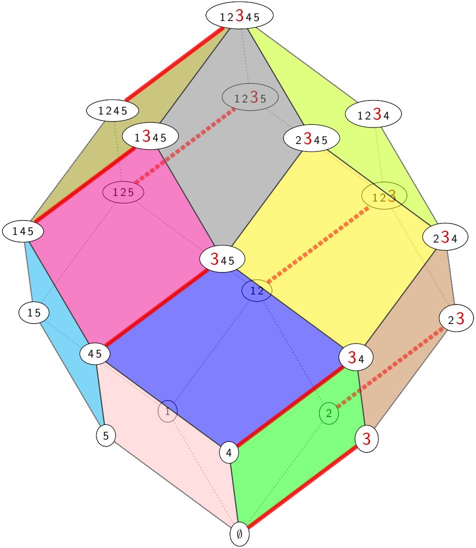
Pseudoplane arrangements



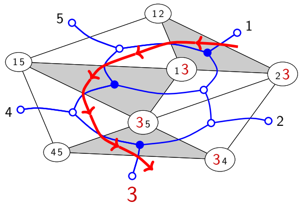


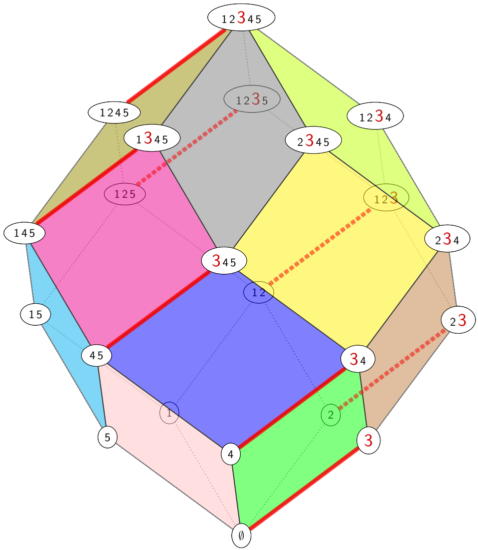
Pseudoplane arrangements





Pseudoplane arrangements





Main result

Theorem (G. (2017))

(k,n)-*plabic graphs* $\xleftrightarrow[\textit{dual}]{\textit{planar}}$ *horizontal sections at* $z = k$ *of fine zonotopal tilings of* $\mathcal{Z}(n,3)$

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Purity $\Longrightarrow$ *every* (k,n)*-plabic graph arises this way*

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Purity $\Longrightarrow$ *every* (k,n)*-plabic graph arises this way*
Moves $=$ *horizontal sections of flips*
Strands $=$ *horizontal sections of pseudoplanes.*

Bibliography

Slides: http://math.mit.edu/~galashin/slides/FPSAC2018.pdf

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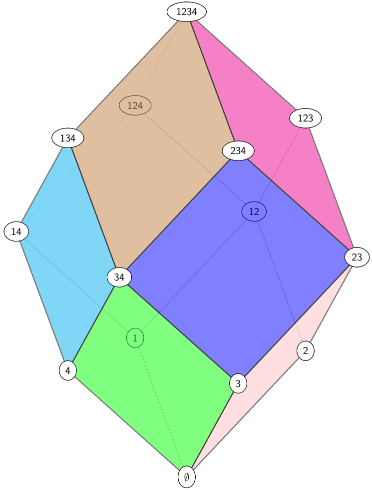
Thank you!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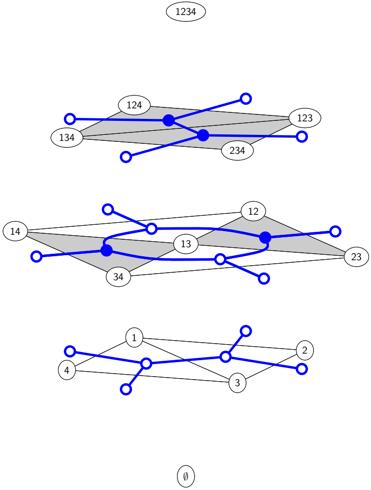
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