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Zamolodchikov periodicity and integrability (extended abstract)

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Abstract. The T-system is a certain discrete dynamical system associated with a quiver. Keller showed in 2013 that the T-system is periodic when the quiver is a product of two finite Dynkin diagrams. We prove that the T-system is periodic if and only if the quiver is a finite $\boxtimes$ finite quiver. Such quivers have been classified by Stembridge in the context of Kazhdan-Lusztig theory. We show that if the T-system is linearizable then the quiver is necessarily an affine $\boxtimes$ finite quiver. We classify such quivers and conjecture that the T-system is linearizable for each of them. For affine $\boxtimes$ finite quivers of type $\hat{A} \otimes A$, that is, for the octahedron recurrence on a cylinder, we give an explicit formula for the linear recurrence coefficients in terms of the partition functions of domino tilings of a cylinder. Next, we show that if the T-system grows slower than a double exponential function then the quiver is an affine $\boxtimes$ affine quiver, and classify them as well. Additionally, we prove that the cube recurrence introduced by Propp is periodic inside a triangle and linearizable on a cylinder.

Résumé. The T-system is a certain discrete dynamical system associated with a quiver. Keller showed in 2013 that the T-system is periodic when the quiver is a product of two finite Dynkin diagrams. We prove that the T-system is periodic if and only if the quiver is a finite $\boxtimes$ finite quiver. Such quivers have been classified by Stembridge in the context of Kazhdan-Lusztig theory. We show that if the T-system is linearizable then the quiver is necessarily an affine $\boxtimes$ finite quiver. We classify such quivers and conjecture that the T-system is linearizable for each of them. For affine $\boxtimes$ finite quivers of type $\hat{A} \otimes A$, that is, for the octahedron recurrence on a cylinder, we give an explicit formula for the linear recurrence coefficients in terms of the partition functions of domino tilings of a cylinder. Next, we show that if the T-system grows slower than a double exponential function then the quiver is an affine $\boxtimes$ affine quiver, and classify them as well. Additionally, we prove that the cube recurrence introduced by Propp is periodic inside a triangle and linearizable on a cylinder.

Keywords: Cluster algebras, Zamolodchikov periodicity, domino tilings, linear recurrence, cube recurrence, commuting Cartan matrices

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1 Introduction

Cluster algebras have been introduced by Fomin and Zelevinsky in [3] and since then have been a popular subject of research. An important class of cluster algebras are those associated with *quivers* which are directed graphs without loops and directed 2-cycles. One can define an operation on quivers called *mutation*: given a quiver Q with vertex set $\mathrm{Vert}(Q)$ and its vertex $v \in \mathrm{Vert}(Q)$, $\mu_v(Q)$ is another quiver on the same set of vertices as Q but with edges modified according to a certain combinatorial rule.

We say that a quiver Q is *bipartite* if the underlying graph is bipartite. In other words, Q is bipartite if there is a map $\epsilon : \mathrm{Vert}(Q) \to \{0,1\}, v \mapsto \epsilon_v$ such that for any arrow $u \to v$ in Q we have $\epsilon_u \neq \epsilon_v$. It is clear from the definition of a mutation that if there are no arrows between u and v then the operations μ_u and μ_v commute. Thus if Q is bipartite, one can define two operations μ_0 and μ_1 on Q as products $\mu_0 = \prod_{\epsilon_u=0} \mu_u$ and $\mu_1 = \prod_{\epsilon_v=1} \mu_v$.

Let us say that Q^{op} is the same quiver as Q but with all edges reversed. Then we call a bipartite quiver Q *recurrent* if $\mu_0(Q) = \mu_1(Q) = Q^{\mathrm{op}}$. In other words, a bipartite quiver Q is recurrent if mutating all vertices of the same color reverses the arrows of Q but does not introduce any new arrows. We restate this definition in an elementary way in Section 3. The notion of recurrent quivers is necessary to define the *T-system* which we do now. For a quiver Q let $\mathbf{x} = \{x_v\}_{v\in\mathrm{Vert}(Q)}$ be a family of indeterminates and let $\mathbb{Q}(\mathbf{x})$ be the field of rational functions in these variables. Then given a bipartite recurrent quiver Q, the *T-system* associated with Q is a family of rational functions $T_v(t) \in \mathbb{Q}(\mathbf{x})$ for each $v \in \mathrm{Vert}(Q)$ and $t \in \mathbb{Z}$ satisfying the following recurrence relation for all $v \in \mathrm{Vert}(Q)$ and all $t \in \mathbb{Z}$:

$$T_v(t+1)T_v(t-1) = \prod_{u\to v} T_u(t) + \prod_{v\to w} T_w(t). \tag{1.1}$$

One immediately observes that the parity of $t+\epsilon_v$ is the same in each term of (1.1) so the T-system splits into two independent parts. Thus we restrict the elements $T_v(t)$ of the T-system to only the values of t for which $t \equiv \epsilon_v \pmod 2$. The initial conditions for the T-system are given by $T_v(\epsilon_v) = x_v, \quad v \in \mathrm{Vert}(Q)$. It is clear that these initial conditions together with (1.1) determine $T_v(t)$ for all $v \in \mathrm{Vert}(Q)$ and $t \equiv \epsilon_v \pmod 2$.

During the past two decades, various special cases of T-systems have been studied extensively, the most popular one being the *octahedron recurrence*. More generally, given two ADE Dynkin diagrams Λ and Λ', one can define their *tensor product* $\Lambda \otimes \Lambda'$ which is a bipartite recurrent quiver, see Figure 1 (a) for an example. For these quivers, the associated T system turns out to be *periodic*, that is, for every ADE Dynkin diagrams Λ and Λ' there is an integer N such that the T-system associated with $\Lambda \otimes \Lambda'$ satisfies $T_v(t) = T_v(t+2N)$ for all $v \in \mathrm{Vert}(Q)$ and $t \equiv \epsilon_v \pmod 2$. This result has been recently shown by Keller [8]. There is also a nice formula for the period N of the T-system

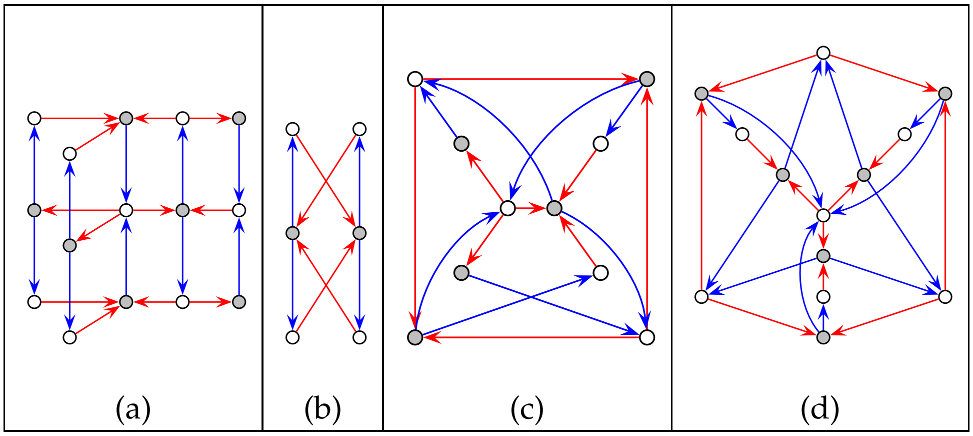

Figure 1: (a) A tensor product $D_5 \otimes A_3$. (b) A finite $\boxtimes$ finite quiver. (c) An affine $\boxtimes$ finite quiver. (d) An affine $\boxtimes$ affine quiver. Arrows are colored according to Definition 2.1.

associated with $\Lambda \otimes \Lambda'$, namely, N divides $h(\Lambda) + h(\Lambda')$ where h denotes the *Coxeter number* of the corresponding Dynkin diagram, see Section 3.

Remark 1.1. The standard formulation of Zamolodchikov periodicity includes *Y-systems* rather than T-systems. However, the machinery of cluster algebras *with principal coefficients* [5] allows one to show that given a bipartite recurrent quiver, the T-system is periodic if and only if the Y-system is periodic.

One other interesting phenomenon related to T-systems has been studied to some extent. Given a bipartite recurrent quiver Q, let us say that the T-system associated with Q is *linearizable* if for every vertex $v \in \mathrm{Vert}(Q)$, there exists an integer N and rational functions $H_0, H_1, \dots, H_N \in \mathbb{Q}(\mathbf{x})$ such that $H_0, H_N \neq 0$ and $\sum_{i=0}^{N} H_i T_v(t+i) = 0$ for every $t \in \mathbb{Z}$ satisfying $t \equiv \epsilon_v \pmod 2$. It was shown in [1] that if every vertex of Q is either a source or a sink and the T-system associated with Q is linearizable then the underlying graph of Q is necessarily an affine ADE Dynkin diagram. Conversely, for every such quiver the T-system was shown to be linearizable in [1, 9].

2 Main results

Before we state our results, we need to define various classes of quivers.

Definition 2.1. Given a bipartite quiver Q with vertex set $\mathrm{Vert}(Q)$, we define two undirected graphs $\Gamma = \Gamma(Q)$ and $\Delta = \Delta(Q)$ on $\mathrm{Vert}(Q)$ as follows. For every arrow $u \to v$ with $\epsilon_u = 0, \epsilon_v = 1$, Γ contains an undirected edge (u, v), and for every arrow $u \to v$ with $\epsilon_u = 1, \epsilon_v = 0$, Δ contains an undirected edge (u, v).

Q **is a**	**if all components of** $\Gamma(Q)$ **are**	**and all components of** $\Delta(Q)$ **are**
finite $\boxtimes$ finite quiver	finite ADE Dynkin diagrams	finite ADE Dynkin diagrams
affine $\boxtimes$ finite quiver	affine ADE Dynkin diagrams	finite ADE Dynkin diagrams
affine $\boxtimes$ affine quiver	affine ADE Dynkin diagrams	affine ADE Dynkin diagrams

Table 1: Three classes of bipartite recurrent quivers

Thus each arrow of Q belongs either to Γ or to Δ. Just as in Figure 1, if an arrow belongs to Γ then we color it red, otherwise we color it blue. Since Q is allowed to have multiple arrows, Γ and Δ can have multiple edges but no loops.

We now define *finite $\boxtimes$ finite, affine $\boxtimes$ finite, and affine $\boxtimes$ affine* quivers. The three definitions are encoded in Table 1. For example, a bipartite recurrent quiver Q is called *affine $\boxtimes$ finite* if all components of $\Gamma(Q)$ are affine ADE Dynkin diagrams and all components of $\Delta(Q)$ are finite ADE Dynkin diagrams. Finite $\boxtimes$ finite quivers appeared in Stembridge's study [12] of *admissible W-graphs* where he showed that they correspond to pairs of (possibly non-reduced) commuting Cartan matrices. He gave a classification of such quivers, and an example of a finite $\boxtimes$ finite quiver is shown in Figure 1 (b).

Note that tensor products $\Lambda \otimes \Lambda'$ of finite ADE Dynkin diagrams belong to the class of finite $\boxtimes$ finite quivers. Our first result combined with Stembridge's work gives a classification of bipartite recurrent quivers for which the T-system is periodic:

Theorem 2.2 ([**?**])**.** *Let Q be a bipartite recurrent quiver. Then the T-system associated with Q is periodic if and only if Q is a finite $\boxtimes$ finite quiver.*

Remark 2.3. Even though this is a generalization of Keller's theorem, we use his result in our proof. Hence we do not give an alternative proof of periodicity for products of finite ADE Dynkin diagrams. We do however give an alternative proof for quivers of type $A_m \otimes A_n$ in [**?**, Section 8] thus reproving Volkov's result [13].

In fact, the following proposition which is due to Stembridge provides a way to extend Keller's formula for the period of the T-system:

Proposition 2.4 ([12])**.** *Let Q be a finite $\boxtimes$ finite bipartite recurrent quiver. Then all connected components of $\Gamma(Q)$ (resp., of $\Delta(Q)$) have the same Coxeter number denoted $h(Q)$ (resp., $h'(Q)$).*

Theorem 2.5 ([**?**])**.** *For a finite $\boxtimes$ finite quiver Q, the period N of the T-system divides $h(Q) + h'(Q)$.*

We now pass to the linearizability property of the T-system.

Theorem 2.6 ([6])**.** *Let Q be a bipartite recurrent quiver, and suppose that the T-system associated with Q is linearizable. Then Q is an affine $\boxtimes$ finite quiver.*

We have classified affine $\boxtimes$ finite quivers, see Figure 1 (c) for an example. Computational evidence suggests the following

Conjecture 2.7 ([6]). *Let Q be a bipartite recurrent quiver. Then the T-system associated with Q is linearizable if and only if Q is an affine $\boxtimes$ finite quiver.*

When Q is a tensor product of type $\hat{A} \otimes A$, Conjecture 2.7 was proven by the second author in [11]. We extend this result by giving an explicit formula for the recurrence coefficients, see Section 5. We also state and prove periodicity and linearizability for the *cube recurrence*, see Section 6.

For affine $\boxtimes$ affine quivers, we have proven the following theorem concerning the asymptotics of the T-system:

Theorem 2.8 (Galashin-Pylyavskyy(2016)). *Let Q be a bipartite recurrent quiver that is* not *an affine $\boxtimes$ affine quiver. Then for any vertex $v \in \operatorname{Vert}(Q)$ and any map $\lambda : \operatorname{Vert}(Q) \to \mathbb{R}_{>0}$ there exists a positive constant C such that for any $t \equiv \epsilon_v \pmod 2$,*

$$T_v(t) \mid_{\mathbf{x}=\lambda} > C \exp(\exp(C|t|)).$$

Here $T_v(t) \mid_{\mathbf{x}=\lambda}$ means substituting $x_u := \lambda(u)$ into $T_v(t)$ for all $u \in \operatorname{Vert}(Q)$.

We have classified affine $\boxtimes$ affine quivers, an example of an affine $\boxtimes$ affine quiver is shown in Figure 1 (d). To sum up: if Q is a finite $\boxtimes$ finite quiver then by Theorem 2.2, $T_v(t)$ is bounded; otherwise if Q is an affine $\boxtimes$ finite quiver then Conjecture 2.7 implies that $T_v(t)$ grows exponentially; finally, according to our computations, the following conjecture seems to hold:

Conjecture 2.9. *If Q is an affine $\boxtimes$ affine quiver then $T_v(t)$ grows quadratic exponentially, that is, for any $v \in \operatorname{Vert}(Q)$ and any $\lambda : \operatorname{Vert}(Q) \to \mathbb{R}_{>0}$ there exist two constants $0 < C < C'$ such that $C \exp(Ct^2) \leq T_v(t) \mid_{\mathbf{x}=\lambda} < C' \exp(C't^2)$.*

Theorem 2.10 (Galashin-Pylyavskyy(2016)). *Conjecture 2.9 holds when Q is a tensor product of type $\hat{A}_{2n-1} \otimes \hat{A}_{2m-1}$.*

3 Background

We use the common conventions for finite and affine Dynkin diagrams, for example, A_5 is a path on 5 vertices and $\hat{A}_5$ is a cycle on six vertices. Since we restrict our attention to bipartite quivers, we will consider cycles of even length, that is, of type $\hat{A}_{2n-1}$ for $n \geq 1$.

Every finite ADE Dynkin diagram Λ corresponds to a *finite Coxeter system* (W, S). All the *Coxeter elements* in (W, S) are conjugate and thus have the same period which is denoted $h(\Lambda)$. For example, Dynkin diagram of type A_n corresponds to the symmetric

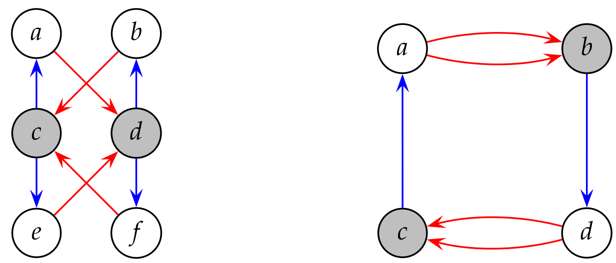

Figure 2: A twist of type $A_3 \times A_3$ (left). A tensor product $\hat{A}_1 \otimes A_2$ (right).

group S_{n+1} where the Coxeter element is just the long cycle $(1, 2, \dots, n, n+1)$ which has period $n+1$, thus $h(A_n) = n+1$. Similarly, $h(D_n) = 2n-2$, $h(E_6) = 12$, $h(E_7) = 18$, and $h(E_8) = 30$.

Suppose we are given two bipartite undirected graphs Λ, Λ'. Let $\epsilon : \mathrm{Vert}(\Lambda) \to \{0, 1\}$ and $\epsilon' : \mathrm{Vert}(\Lambda') \to \{0, 1\}$ be the corresponding colorings of their vertices. Then we define the *tensor product* $\Lambda \otimes \Lambda'$ to be the quiver Q with vertex set $\mathrm{Vert}(\Lambda) \times \mathrm{Vert}(\Lambda')$ and edges given by the following rule: $(u, u') \to (v, u')$ is an edge of Q if (u, v) is an edge of Λ and $(\epsilon_u, \epsilon'_{u'}) \in \{(0, 0), (1, 1)\}$; similarly, $(u, u') \to (u, v')$ is an edge of Q if (u', v') is an edge of Λ' and $(\epsilon_u, \epsilon'_{u'}) \in \{(0, 1), (1, 0)\}$. In other words, $\Lambda \otimes \Lambda'$ can be described as follows: its underlying undirected graph is just the direct product of Λ and Λ', the red arrows are given by $\Gamma = \Lambda \times \mathrm{Vert}(\Lambda')$ and the blue arrows are given by $\Delta = \mathrm{Vert}(\Lambda) \times \Lambda'$. Note that the rule in Definition 2.1 allows one to reconstruct the directions of the arrows in Q from their colors (Γ, Δ).

Instead of explaining the definition of a quiver mutation, we give an equivalent definition of a recurrent quiver: a bipartite quiver Q is called *recurrent* if for any two vertices $u, v \in \mathrm{Vert}(Q)$, the number of paths $u \to w \to v$ of length 2 from u to v equals the number of paths $v \to w \to u$ of length 2 from v to u. Yet another equivalent way of giving this definition is that Q is recurrent if the adjacency matrices of $\Gamma(Q)$ and $\Delta(Q)$ commute.

4 Zamolodchikov periodicity: an example

In this section, we give an example illustrating Theorems 2.2 and 2.5. Consider the quiver Q which Stembridge [12] called a *twist of type* $A_3 \times A_3$. It has six vertices which we label a, b, c, d, e, f, see Figure 2 (left).

Let us plug in $x_a = 3, x_d = 2$ and $x_b = x_c = x_e = x_f = 1$ for simplicity. The T-system associated with Q proceeds according to equation (1.1), for example, $T_a(t+1)T_a(t-1) = T_c(t) + T_d(t)$ and $T_c(t+1)T_c(t-1) = T_a(t)T_e(t) + T_b(t)T_f(t)$. We list the values $T_v(t)$ for all $t = 0, 1, \dots, 9$ in Table 2. For example, we got $T_c(5) = \frac{T_a(4)T_e(4)+T_b(4)T_f(4)}{T_c(3)} = \frac{18\times 6+6\times 6}{12} = 12$.

t		0		1		2		3		4		5		6		7		8		9	
$T_a(t)$	$T_b(t)$	3	1	$*$	$*$	1	3	$*$	$*$	18	6	$*$	$*$	2	6	$*$	$*$	3	1	$*$	$*$
$T_c(t)$	$T_d(t)$	$*$	$*$	1	2	$*$	$*$	12	6	$*$	$*$	12	24	$*$	$*$	4	2	$*$	$*$	1	2
$T_e(t)$	$T_f(t)$	1	1	$*$	$*$	3	3	$*$	$*$	6	6	$*$	$*$	6	6	$*$	$*$	1	1	$*$	$*$

Table 2: The values of the T-system associated with a twist of type $A_3 \times A_3$. For $t \not\equiv \epsilon_v \pmod 2$, $T_v(t)$ is undefined so we replace it with a $*$.

We can see that $T_v(t) = T_v(t+8)$ for each $t \equiv \epsilon_v \pmod 2$ so the period $N = 4$ indeed divides $h(A_3) + h(A_3) = 4 + 4 = 8$ as predicted by Theorem 2.5.

5 Domino tilings

Here we explain our formula for the recurrence coefficients of the T-system associated with a quiver Q which is a tensor product of type $\hat{A}_{2n-1} \otimes A_m$, that is, a product of a $2n$-cycle and an m-path. Consider an $(m+1) \times 2n$ cylinder C. The vertices of Q can be naturally identified with the interior vertices of C. For example, if $n = 1$ and $m = 2$ then Q has vertices a, b, c, d as in Figure 2 (right) and it is embedded in the 3×2 cylinder in Figure 3 (left). For every domino tiling T of C we define a monomial $\mathrm{wt}(T)$ in the initial variables $\mathbf{x}$ and an integer $\mathrm{ht}(T)$ called the *height* of T as follows. For every vertex $v \in \mathrm{Vert}(Q)$, define an integer $\mathrm{adj}_T(v)$ to be the number of dominoes adjacent to v in T, so $2 \le \mathrm{adj}_T(v) \le 4$. Then we set $\mathrm{wt}(T) = \prod_{v \in \mathrm{Vert}(Q)} x_v^{\mathrm{adj}_T(v)-3}$. Various domino tilings with their weights are listed in Figures 3 and 4.

To define $\mathrm{ht}(T)$, we will use an auxiliary function h_T defined on the vertices of C analogously to the *Thurston height* function for planar domino tilings. Namely, fix some vertex s on the bottom boundary of C and set $h_T(s) = 0$. Then for every edge $e = (u, v)$ of C that does not cut through a domino of T, the face that appears to the left when we traverse e from u to v is either black or white. If it is black then we put $h_T(v) = h_T(u) - 1$, otherwise we put $h_T(v) = h_T(u) + 1$. It is not hard to see that this produces a well defined function h_T which takes values 0 and 1 on the bottom boundary of C and some values x and $x + 1$ on the top boundary of C. In this case, we put $\mathrm{ht}(T) := x$.

There are unique domino tilings T_{-1} and T_1 with $\mathrm{ht}(T_{-1}) = -2 - 2m$ and $\mathrm{ht}(T_{+1}) = 2 + 2m$, and they satisfy $\mathrm{wt}(T_{-1}) = \mathrm{wt}(T_1) = 1$. Figure 3 (middle and right) contains T_1 and T_{-1} together with their corresponding height functions h_{T_1} and $h_{T_{-1}}$. Thus all tilings T of C satisfy $\mathrm{ht}(T) = -2 - 2m + 4i$ for some $0 \le i \le m + 1$. We define Laurent polynomials H_i for $0 \le i \le m+1$ as the corresponding partition functions: $H_i = \sum \mathrm{wt}(T)$ where the sum is taken over all tilings T of C with $\mathrm{ht}(T) = -2 - 2m + 4i$. We call them *Goncharov-Kenyon Hamiltonians* as they have been introduced and studied by Goncharov

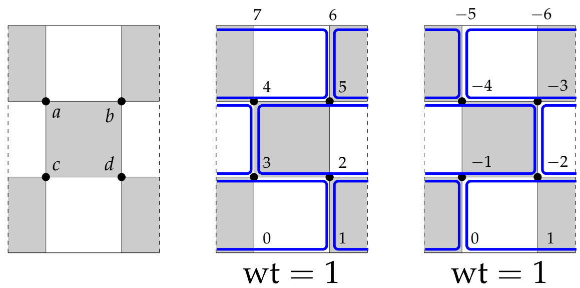

Figure 3: A 3×2 cylinder C (left). The unique domino tilings T_1 of height 6 (middle) and T_{-1} of height -6 (right).

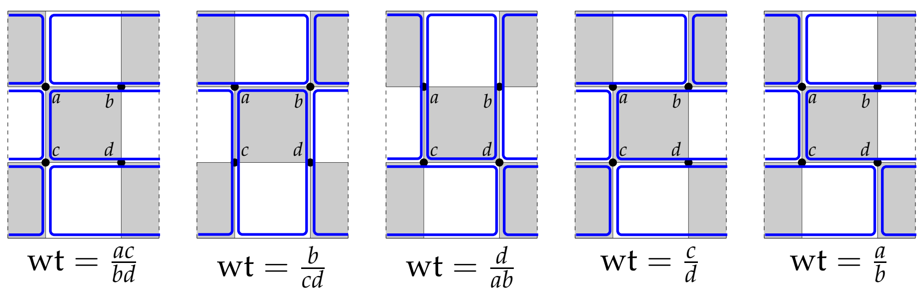

Figure 4: All five domino tilings of height 2.

and Kenyon [7] in a somewhat different context.

For our running example $n = 1, m = 2$, we have already seen that $H_0 = H_3 = 1$. All five tilings with $\mathrm{ht}(T) = 2$ are shown in Figure 4. All five tilings with $\mathrm{ht}(T) = -2$ are just their mirror images. This yields $H_1 = \frac{bd}{ac} + \frac{a}{cd} + \frac{c}{ab} + \frac{d}{c} + \frac{b}{a}$ and $H_2 = \frac{ac}{bd} + \frac{b}{cd} + \frac{d}{ab} + \frac{c}{d} + \frac{a}{b}$.

For a vertex $v \in \mathrm{Vert}(Q)$, define $v_+ := v$ and v_- to be the vertex opposite to v in the same red connected component as v (which is a cycle of length $2n$). Thus for the quiver in Figure 2 (right), we have $a_- = b, b_- = a, c_- = d, d_- = c$. We are finally ready to state the formula for the recurrence:

Theorem 5.1 ([6])**.** *Let v be a vertex on the top boundary of a quiver Q of type $\hat{A}_{2n-1} \otimes A_m$. Then for any $t \in \mathbb{Z}$ with $t \equiv \epsilon_v \pmod 2$, the values of the T-system associated with Q satisfy*

$$T_{v_+}(t + (m+1)n) - H_1 T_{v_-}(t + mn) + \ldots \pm H_m T_{v_\pm}(t+n) \mp T_{v_\mp}(t) = 0. \tag{5.1}$$

Theorem 5.1 only gives a formula when v belongs to the boundary of the cylinder. If v is distance r from the boundary then we prove a formula that looks similar to (5.1) except that now j-th coefficient is the image of the symmetric polynomial $e_j[e_r]$ under the ring homomorphism that sends e_i to H_i for $i = 0, 1, \ldots, m+1$.

As an illustration, let us plug in $x_a = x_b = x_c = x_d = 1$ and run the T-system. One easily checks that the sequence y_n equal to $T_a(n)$ when n is even and to $T_b(n)$ when n

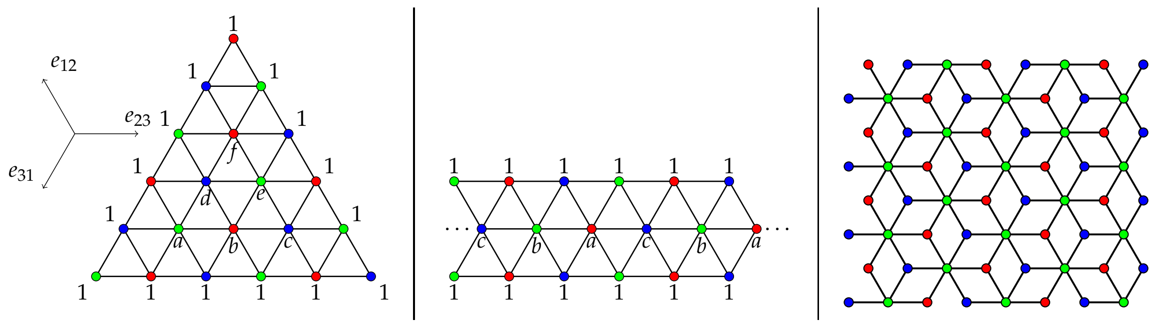

Figure 5: The triangle $\mathbb{T}_5$ (left). The cylinder $\mathbb{S}_2$ for $n = 1$ (middle). The graph G (right). The red, green, and blue colors correspond to $\epsilon_v = 0, 1, 2$ respectively.

is odd satisfies $y_0 = y_1 = 1$ and $y_{n+1} = \frac{y_n^2 + y_n}{y_{n-1}}$ for $n \geq 1$. The first few values of y_n are therefore $1, 1, 2, 6, 21, 77, 286, \ldots$. On the other hand, we have $H_0 = H_3 = 1$ and $H_1 = H_2 = 5$ so Theorem 5.1 suggests that the values of x_n satisfy $y_{n+3} - 5y_{n+2} + 5y_{n+1} - y_n = 0$ for all n. This is indeed true, for example, $77 - 5 \times 21 + 5 \times 6 - 2 = 0$.

6 Cube recurrence

6.1 Periodicity in a triangle

For any $m \in \mathbb{Z}$, let $\mathbb{P}_m = \{(i, j, k) \in \mathbb{Z}^3 \mid i + j + k = m\}$ be a plane. Given an integer $m \geq 3$, we define the m-th *triangle* $\mathbb{T}_m \subset \mathbb{Z}^3$ by $\mathbb{T}_m = \{(i, j, k) \in \mathbb{P}_m \mid i, j, k \geq 0\}$. For example, $\mathbb{T}_5$ is shown in Figure 5 (left). For every vertex $v = (i, j, k) \in \mathbb{T}_m$, we define its *color* $\epsilon_v \in \{0, 1, 2\}$ by $\epsilon_v \equiv j - k \pmod 3 \in \{0, 1, 2\}$. We refer to v as a *boundary vertex* if either one of i, j, k is zero. For every non-boundary vertex v we introduce a variable x_v and we let $\mathbf{x}$ be the set of all these variables. We consider an analog of the T-system in a triangle which is going to be a family $f_v(t)$ of rational functions in $\mathbf{x}$ defined whenever $t \equiv \epsilon_v \pmod 3$. Let $e_{12} = (1, -1, 0), e_{23} = (0, 1, -1)$, and $e_{31} = (-1, 0, 1)$ be three vectors in $\mathbb{P}_0$. For boundary vertices v we set $f_v(t) = 1$ and for every non-boundary vertex $v = (i, j, k) \in \mathbb{T}_m$ we set $f_v(\epsilon_v) = x_v$. For every such v and every $t \equiv \epsilon_v \pmod 3$, f_v satisfies

$$f_v(t+3)f_v(t) = f_{v+e_{12}}(t+2)f_{v-e_{12}}(t+1) + f_{v+e_{23}}(t+2)f_{v-e_{23}}(t+1) + f_{v+e_{31}}(t+2)f_{v-e_{31}}(t+1).$$

The *unbounded cube recurrence*, i.e. the one defined on $\mathbb{P}_0$ rather than $\mathbb{T}_m$, was introduced by Propp [10] where he conjectured that the values are Laurent polynomials. This was proven by Fomin-Zelevinsky [4], and Carroll and Speyer [2] later gave an explicit formula for them in terms of *groves*.

t	$0,1,2$	3	4	5	6	7	8	9	$10,11,12$
$f_f(t)$ $f_d(t)$ $f_e(t)$ $f_a(t)$ $f_b(t)$ $f_c(t)$	1 1 1 1 1 3	3 * * * 5 *	* * 15 7 * *	* 41 * * * 7	19 * * * 21 *	* * 13 9 * *	* 5 * * * 5	1 * * * 3 *	3 1 1 1 1 1

Table 3: The evolution of the cube recurrence in $\mathbb{T}_5$.

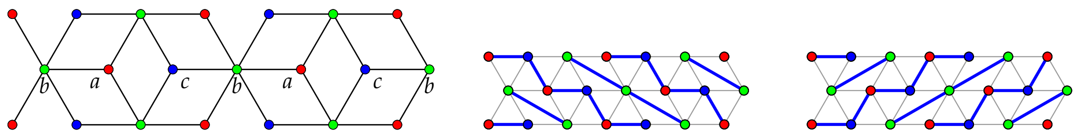

Figure 6: The graph G_2 (left). The unique groves with $h = 0$ (middle) and $h = 2$ (right).

Theorem 6.1 (Galashin-Pylyavskyy(2016))**.** *The values of the cube recurrence in a triangle $\mathbb{T}_m$ are Laurent polynomials. Moreover, let $\sigma : \mathbb{T}_m \to \mathbb{T}_m$ be the clockwise rotation of $\mathbb{T}_m$ defined by $\sigma(i,j,k) = (k,i,j)$. Then for every $v \in \mathbb{T}_m$ and every $t \equiv \epsilon_v \pmod 3$, we have $f_v(t+2m) = f_{\sigma^m v}(t)$. Thus the cube recurrence in a triangle satisfies $f_v(t+6m) = f_v(t)$.*

We give two proofs for Theorem 6.1, one based on Henriques and Speyer's *multidimensional cube recurrence* and one similar to our proof of Theorem 2.2 using a tropicalization argument.

Let us illustrate Theorem 6.1 by an example for $m = 5$. Suppose we set $f_c(\epsilon_c) = 3$ and $f_v(\epsilon_v) = 1$ for $v = a,b,d,e,f$. Then the values of $f_v(t)$ for $t = 0,1,\dots,12$ are shown in Table 3. For example, $f_e(7) = \frac{f_f(6)f_c(5)+f_b(6)+f_d(5)}{f_e(4)} = \frac{19\times 7+21+41}{15} = 13$. Just as Theorem 6.1 states, increasing t by 10 corresponds to rotating the triangle counterclockwise which is the same as applying σ five times.

6.2 Linearizability on a cylinder

We define the *cube recurrence on a cylinder* as follows. Let $m \geq 2, n \geq 1$ be two integers and define the strip $\mathbb{S}_m = \{(i,j,k) \in \mathbb{P}_0 \mid 0 \leq i \leq m\}$. We let g be the vector $ne_{23} = (0,n,-n)$, and everything in this section will be invariant with respect to the shift by $3g$. For every $v = (i,j,k) \in \mathbb{S}_m$ with $0 < i < m$, we introduce a variable x_v so that $x_v = x_{v+3g}$ and we define the cube recurrence on a cylinder to be a family $f_v(t)$ for $v \in \mathbb{S}_m$ that satisfies

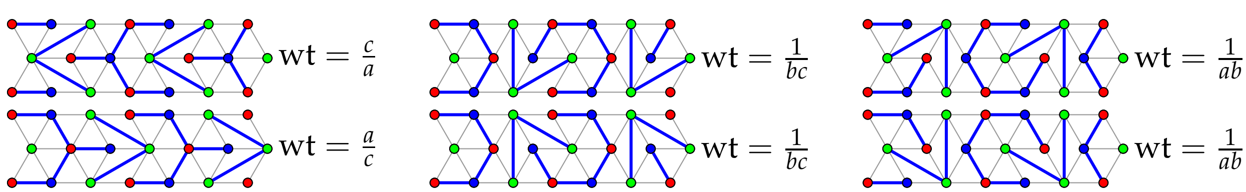

Figure 7: The six $(3,2)$-groves satisfying $h = 1$ together with their weights.

the same recurrence as before but subject to different boundary conditions: $f_v(t) = 1$ whenever $i = 0$ or $i = m$ and $f_v(\epsilon_v) = x_v$ for all $v \in \mathbb{S}_m$.

Theorem 6.2 (Galashin-Pylyavskyy(2016))**.** *Fix any n and m and let $v \in \mathbb{S}_m$ be a vertex. Then the sequence $(f_v(\epsilon_v + 3t))_{t\in\mathbb{Z}}$ satisfies a linear recurrence.*

We also give an explicit formula for the recurrence coefficients when $v = (1, j, k)$ for some j and k. Consider the following infinite undirected graph G with vertex set $\mathbb{P}_0$ and edge set consisting of edges $(v, v + e_{12})$, $(v, v + e_{23})$, and $(v, v + e_{31})$ for every vertex $v \in \mathbb{P}_0$ with $\epsilon_v \neq 0$, see Figure 5 (right).

We let G_m be the restriction of G to $\mathbb{S}_m$, thus G_m is a graph on a strip with vertex set $\mathbb{S}_m$ whose faces are all either lozenges or boundary triangles, see Figure 6 (left). A $(3n, m)$-*grove* is a forest F with vertex set $\mathbb{S}_m$ satisfying several conditions. First, F has to be invariant under the shift by $3g$. Second, F necessarily contains all edges $(v, v + e_{23})$ for boundary vertices v with $\epsilon_v = 0$. Third, for every lozenge face of G_m, F contains exactly one of its two diagonals. And finally, every connected component of F has to contain a vertex $(0, j, k)$ and a vertex (m, j', k') for some $j, j', k, k' \in \mathbb{Z}$.

For $v \in \mathbb{S}_m$ and a $(3n, m)$-grove F, define $\deg_F(v)$ to be the number of edges of F incident to v. Define the *weight* of F to be $\mathrm{wt}(F) = \prod x_v^{\deg_F(v)-2}$ where the product is taken over all non-boundary vertices $v = (i, j, k)$ of $\mathbb{S}_m$ satisfying $0 \leq j < 3n$. The second condition in the definition of a grove together with the construction of G_m implies that every connected component of F involves either only vertices v with $\epsilon_v = 1$ (we call such components *green* because in our figures the green color corresponds to $\epsilon_v = 1$) or only vertices v with $\epsilon_v \neq 1$. Consider any green connected component C of F. Given such C, the unique green lower boundary vertex of C is $u(C) = (0, j, -j)$ for some $j \equiv 2 \pmod 3$, and there is a unique green upper boundary vertex $w(C) = (m, j', k')$. The possible values of j' are $j - 2m, j - 2m + 3, \ldots, j + m$. We define $h(C) := (j' - j + 2m)/3 \in \{0, 1, \ldots, m\}$, and it is clear that this number is the same for any green connected component of F. We define $h(F)$ to be equal to $h(C)$ where C is any such connected component of F. Finally, for $\ell = 0, 1, \ldots, m$, we define $J_\ell := \sum_F \mathrm{wt}(F)$ where the sum is taken over all groves F with $h(F) = \ell$. As it is clear from Figure 6 (middle and right), for $\ell = 0$ or $\ell = m$ there is only one grove F with $h(F) = \ell$ and it satisfies $\mathrm{wt}(F) = 1$, thus $J_0 = J_m = 1$.

Theorem 6.3 (Galashin-Pylyavskyy(2016))**.** *Fix any n and m and let $v = (1, j, k) \in \mathbb{S}_m$. Then for any $t \equiv \epsilon_v \pmod 3$ we have $\sum_{\ell=0}^{m} (-1)^\ell J_\ell f_{v+\ell g}(t + 2\ell n) = 0$.*

For example, let $m = 2$. Then $J_0 = J_2 = 1$, and all the six groves with $h(F) = 1$ are shown in Figure 7 which implies that $J_1 = \frac{c}{a} + \frac{a}{c} + \frac{2}{bc} + \frac{2}{ab}$. Let us plug in $x_v = 1$ for $v = a, b, c$. Then the sequence $(y_n) = (f_a(0), f_b(1), f_c(2), f_a(3), \ldots)$ satisfies $y_0 = y_1 = y_2 = 1$ and $y_{n+3} = \frac{y_{n+2}y_{n+1}+2}{y_n}$, so the first few values are $1, 1, 1, 3, 5, 17, 29, 99, 169 \ldots$. Theorem 6.3 states that $y_{n+4} - 6y_{n+2} + y_n = 0$ for all n which is indeed true, for example, $99 - 6 \times 17 + 3 = 0$.

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