Research Statement

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My primary area of research is algebraic combinatorics, with a focus on total positivity, cluster algebras, and the interplay between them. I am particularly interested in the topology of totally positive spaces, their appearances in physics and statistical mechanics, and discrete integrable systems arising from cluster algebras. Some of these research directions are briefly reviewed below.

1 Totally positive spaces

One of the most well-studied examples of a totally positive space is the *totally nonnegative Grassmannian* $\mathrm{Gr}_{\geq 0}(k,n)$ introduced by Postnikov [Pos06]. This is the space of all $k\times n$ real matrices of rank k viewed modulo row operations, and such that all of their maximal minors are nonnegative. A fundamental question one can ask about such a space is: what is the topology of $\mathrm{Gr}_{\geq 0}(k,n)$? Together with Steven Karp and Thomas Lam, we proved the following result.[1]

Theorem 1.1 ([1])**.** *The space* $\mathrm{Gr}_{\geq 0}(k,n)$ *is homeomorphic to a* $k(n-k)$*-dimensional closed ball.*

This result was conjectured by Postnikov in 2006. He also gave a decomposition of $\mathrm{Gr}_{\geq 0}(k,n)$ into *positroid cells* according to which of the minors are zero and which are positive, and conjectured that the closure of each cell is also homeomorphic to a ball. We give a proof of this conjecture in our forthcoming paper [3].

Theorem 1.2 ([3])**.** *The closure of each positroid cell is homeomorphic to a closed ball.*

Since 2006, much progress has been made towards the proof of Theorem 1.2. It was shown by Williams [Wil07] that the face poset of this cell decomposition is shellable, and then Postnikov–Speyer–Williams [PSW09] showed that $\mathrm{Gr}_{\geq 0}(k,n)$ is a CW complex. Later, Rietsch–Williams [RW10] proved that the closure of each cell is contractible.

Our proof of Theorem 1.1 is completely elementary and involves the *cyclic shift vector field* that contracts the whole space $\mathrm{Gr}_{\geq 0}(k,n)$ to the unique *cyclically symmetric point* $X_0\in\mathrm{Gr}_{\geq 0}(k,n)$, that is, the unique element of $\mathrm{Gr}_{\geq 0}(k,n)$ invariant under cyclically shifting the columns of the corresponding $k\times n$ matrix. On the other hand, the proof of Theorem 1.2 relies on some heavy machinery such as the *generalized Poincaré conjecture* due to Smale, Freedman, and Perelman, combined with affine flag variety analogs of *Fomin–Shapiro projections* [FS00].

Separately, Lusztig [Lus94, Lus98] introduced the totally nonnegative part $(G/P)_{\geq 0}$ of a partial flag variety G/P in a reductive algebraic group G as an application of his theory of *canonical bases* [Lus90]. He showed [Lus98] that $(G/P)_{\geq 0}$ is contractible, and conjectured a cell decomposition of this space. Later, Rietsch proved that Postnikov's $\mathrm{Gr}_{\geq 0}(k,n)$ is a special case of Lusztig's $(G/P)_{\geq 0}$, and that the corresponding cell decompositions coincide. In [2], we gave a common generalization of Theorem 1.1 and Lusztig's contractibility result.

Theorem 1.3 ([2])**.** *The space* $(G/P)_{\geq 0}$ *is homeomorphic to a closed ball.*

Even for $G=\mathrm{GL}_n(\mathbb{R})$, this theorem includes some new important special cases such as the totally nonnegative part of the *complete flag variety* $(\mathrm{GL}_n(\mathbb{R})/B)_{\geq 0}$. The ultimate goal of this ongoing project is to generalize the result of Theorem 1.2 to all $(G/P)_{\geq 0}$, proving the following conjecture of [Wil07].

Conjecture 1.4. *The closure of every cell in the cell decomposition of* $(G/P)_{\geq 0}$ *is homeomorphic to a closed ball.*

An analogous result for the totally nonnegative part of the *unipotent radical* of G was conjectured by Fomin–Shapiro [FS00] and proved by Hersh [Her14]. The cell decomposition of $(G/P)_{\geq 0}$ was shown to be a CW complex by Rietsch and Williams [RW08], who also proved that the closure of every cell is contractible [RW10].

[1] Numbered citations refer to publications and preprints that I coauthored.

2 The Ising model

Totally positive spaces have attracted much attention in particular due to their appearances in other contexts, such as the study of scattering amplitudes [AHT14] and KP solitons [KW14]. Pavlo Pylyavskyy and I have recently discovered [4] a relationship between the totally nonnegative Grassmannian and the *Ising model.* Given a weighted graph G, the Ising model is a probability distribution on the space of *spin configurations* on the vertices of G. This model of ferromagnetism has been studied for more than 100 years, mostly in the setting where G is a planar graph. In this case, recent breakthroughs of Smirnov et al. show that the scaling limit *at critical temperature* exhibits universality and conformal invariance. Given a planar graph G embedded in a disk D, define the *boundary correlation matrix* whose (i,j)-th entry equals the correlation between the spins at vertices i and j located on the boundary of D.

Theorem 2.1 ([4])**.** *There exists an explicit homeomorphism ϕ between the space Γ_n of $n\times n$ boundary correlation matrices and the* totally nonnegative orthogonal Grassmannian $\mathrm{OG}_{\geq 0}(n,2n)$*. Each of the spaces Γ_n and $\mathrm{OG}_{\geq 0}(n,2n)$ is homeomorphic to an $\binom{n}{2}$-dimensional closed ball.*

The space $\mathrm{OG}_{\geq 0}(n,2n)$, introduced recently in the study of *ABJM scattering amplitudes*, is the subset of $\mathrm{Gr}_{\geq 0}(n,2n)$ where each maximal minor is equal to the "complementary" maximal minor. Theorem 2.1 describes the space Γ_n by algebraic inequalities, answering a question of [KS68] in the planar case. The cyclic shift operator on $\mathrm{Gr}_{\geq 0}(n,2n)$ translates under the map ϕ into the fundamental *Kramers–Wannier's duality* [KW41] which was originally used to compute the critical temperature of the Ising model. The unique *cyclically symmetric point* $X_0\in \mathrm{Gr}_{\geq 0}(n,2n)$ admits an interpretation as the point corresponding to the planar Ising model at critical temperature.

Our construction gives rise to many intriguing questions. Most notably, it suggests that studying asymptotic properties of objects associated with total positivity (e.g., the point $X_0\in \mathrm{Gr}_{\geq 0}(n,2n)$ and Postnikov's *plabic graphs*) may lead to direct applications to the scaling limit of the Ising model.

3 The amplituhedron

Our original motivation with Karp and Lam that ultimately led to the proof of Theorem 1.1 was to study the *amplituhedron*, which is a certain subset of the Grassmannian. It was introduced by Arkani-Hamed and Trnka [AHT14] in order to give a geometric basis for computing scattering amplitudes in $\mathcal{N}=4$ supersymmetric Yang–Mills theory. The amplituhedron $\mathcal{A}_{n,k,m}(Z)$ depends on a choice of four integers k,ℓ,m,n such that m is even and $k+\ell+m=n$, and on an $n\times(k+m)$ matrix Z with positive maximal minors. Specifically, $\mathcal{A}_{n,k,m}(Z)$ is just the image of $\mathrm{Gr}_{\geq 0}(k,n)$ under the linear map induced by Z. For $k=1$, $\mathcal{A}_{n,k,m}(Z)$ is a *cyclic polytope.*

In order to compute the scattering amplitude (i.e., the probability that a certain particle interaction occurs), one needs to find a certain differential form ω_{tree}. There are various ways to compute ω_{tree}, e.g., one can use Feynman diagrams, or a much more efficient way is to use the *BCFW recurrence* of [BCFW05]. In either case, there is a large amount of cancellation, which at the end produces a short answer. What Arkani-Hamed and Trnka realized was that the various ways to perform the BCFW recurrence correspond to the various *triangulations* of the amplituhedron $\mathcal{A}_{n,k,m}(Z)$.

The pieces of triangulations of $\mathcal{A}_{n,k,m}(Z)$ are indexed by *affine permutations*, which are bijections $f:\mathbb{Z}\to\mathbb{Z}$ satisfying $f(i+n)=f(i)+n$ for all $i\in\mathbb{Z}$. It is very hard to tell when a collection of affine permutations yields a triangulation of $\mathcal{A}_{n,k,m}(Z)$, and it is in general not even known whether $\mathcal{A}_{n,k,m}(Z)$ admits at least one triangulation when $k,m>2$. Nevertheless, in [5], Thomas Lam and I proved a *parity duality* result for triangulations of the amplituhedron.

Theorem 3.1 ([5])**.** *A collection $f_1,\dots,f_r$ of affine permutations gives a triangulation of $\mathcal{A}_{n,k,m}(Z)$ for all Z if and only if the collection of their inverses $f_1^{-1},\dots,f_r^{-1}$ gives a triangulation of $\mathcal{A}_{n,\ell,m}(Z)$ for all Z.*

We also gave a precise relationship between the corresponding differential forms ω_{tree}, thus establishing the parity duality in the strongest possible form.

For example, Theorem 3.1 shows that the triangulations of the $\ell=1$ amplituhedron are identical to the triangulations of the cyclic polytope. Blagojević, Palić, Ziegler, and I proved in [6] that the $\ell=1$ amplituhedron is homeomorphic to a closed ball. In our original paper [1] with Karp and Lam, we showed that the *cyclically* *symmetric amplituhedron* $\mathcal{A}_{n,k,m}(Z_0)$ is homeomorphic to a closed ball, but it remains an open problem to determine the topology of the amplituhedron in the general case.

Triangulations of $\mathcal{A}_{n,k,m}(Z)$ are important for computing scattering amplitudes, and Theorem 3.1 shows that they have deep combinatorial properties. It would be great to understand their structure in more detail. The following particularly important conjecture is widely accepted in the physics literature.

Conjecture 3.2 ([AHT14])**.** *A collection $f_1,...,f_r$ of affine permutations gives a triangulation of $\mathcal{A}_{n,k,m}(Z)$ for* some Z *if and only if it gives a triangulation of $\mathcal{A}_{n,k,m}(Z)$ for* all Z.

4 Cluster algebras

Fomin–Zelevinsky [FZ02] introduced cluster algebras in the early 2000s, aiming to provide a combinatorial framework for Lusztig's canonical bases and total positivity. Coordinate rings of flag varieties and Grassmannians are special cases of cluster algebras, and their totally positive parts are precisely the subsets where all *cluster variables* are positive. Another motivation for cluster algebras came from the *Zamolodchikov periodicity conjecture* [Zam91], which originally appeared in 1991 in the study of the thermodynamic Bethe ansatz.

A *seed* $(Q,\mathbf{x})$ in a cluster algebra is a collection $\mathbf{x}$ of rational functions labeled by the vertices of a *quiver* Q. Given a vertex v of Q, a *seed mutation* μ_v changes Q and the component x_v of $\mathbf{x}$ according to some combinatorial rules, producing a new seed $\mu_v(Q,\mathbf{x})=(Q',\mathbf{x}')$. When Q is bipartite, we say that Q is *recurrent* if it is invariant under the operation $\mu_\bullet\mu_\circ$ of mutating at all white vertices and then mutating at all black vertices. In this case, the *T-system* associated with Q is the discrete dynamical system obtained by iterating the map $\mu_\bullet\mu_\circ$.

Given a pair (Λ,Λ') of ADE Dynkin diagrams, one can construct a bipartite recurrent quiver $\Lambda\otimes\Lambda'$ called their *tensor product*. The (generalized) Zamolodchikov periodicity conjecture states that the T-system associated with $\Lambda\otimes\Lambda'$ is periodic. Some special cases of this conjecture were completed by Frenkel–Szenes [FS95], Fomin–Zelevinsky [FZ03], Volkov [Vol07], and others. It was finally settled in full generality by Keller [Kel13] using cluster categorification. Together with Pylyavskyy, we discovered many other bipartite recurrent quivers exhibiting Zamolodchikov periodicity.

Theorem 4.1 ([7])**.** *Bipartite recurrent quivers such that the associated T-system is periodic are in bijection with pairs of commuting Cartan matrices of finite type.*

Surprisingly, such pairs of Cartan matrices have already been classified by Stembridge in his study of Kazhdan–Lusztig cells. We thus obtained a complete classification of Zamolodchikov periodic quivers. In addition to tensor products of ADE Dynkin diagrams, it includes another 4 infinite families plus 8 exceptional cases.

A natural question arises: what is the behavior of the T-system for a non-periodic bipartite recurrent quiver Q? The bijection in Theorem 4.1 can be extended to associate a pair (A,A') of commuting matrices to any such Q. We say that Q is *of tame type* if both A and A' are Cartan matrices of either finite or affine type, and otherwise we say that Q is *of wild type*. In [8, 9] we classified all quivers of tame type, and showed that quivers of wild type are never *Zamolodchikov integrable*. The latter is a certain algebraic property of the associated T-system. It remains an open problem to prove the converse to this statement.

Conjecture 4.2 ([8, 9])**.** *Every tame bipartite recurrent quiver is Zamolodchikov integrable.*

Together with our classification results, this would give a complete description of the behavior of the T-system for all bipartite recurrent quivers.

For a complete publication list, see http://math.mit.edu/~galashin/papers.html.

References coauthored by Pavel Galashin

[1] P. Galashin, S. N. Karp, and T. Lam. The totally nonnegative Grassmannian is a ball. arXiv:1707.02010.

[2] P. Galashin, S. N. Karp, and T. Lam. The totally nonnegative part of G/P is a ball. arXiv:1801.08953.

[3] P. Galashin, S. N. Karp, and T. Lam. *In preparation*, 2018.

[4] P. Galashin and P. Pylyavskyy. Ising model and the positive orthogonal Grassmannian. arXiv:1807.03282.

[5] P. Galashin and T. Lam. Parity duality for the amplituhedron. arXiv:1805.00600.

[6] P. Blagojević, P. Galashin, N. Palić, and G. M. Ziegler. Some more amplituhedra are contractible. *Selecta Math. (N.S.)*, to appear.

[7] P. Galashin and P. Pylyavskyy. The classification of Zamolodchikov periodic quivers. *Amer. J. Math.*, to appear.

[8] P. Galashin and P. Pylyavskyy. Quivers with subadditive labelings: classification and integrability. arXiv:1606.04878.

[9] P. Galashin and P. Pylyavskyy. Quivers with additive labelings: classification and algebraic entropy. arXiv:1704.05024.

References

[AHT14] N. Arkani-Hamed and J. Trnka. The amplituhedron. *J. High Energy Phys.*, (10):30, 2014.

[BCFW05] R. Britto, F. Cachazo, B. Feng, and E. Witten. Direct proof of the tree-level scattering amplitude recursion relation in Yang-Mills theory. *Phys. Rev. Lett.*, 94(18):181602, 4, 2005.

[FS95] E. Frenkel and A. Szenes. Thermodynamic Bethe ansatz and dilogarithm identities. I. *Math. Res. Lett.*, 2(6):677–693, 1995.

[FS00] S. Fomin and M. Shapiro. Stratified spaces formed by totally positive varieties. *Michigan Math. J.*, 48:253–270, 2000. Dedicated to William Fulton on the occasion of his 60th birthday.

[FZ02] S. Fomin and A. Zelevinsky. Cluster algebras. I. Foundations. *J. Amer. Math. Soc.*, 15(2):497–529, 2002.

[FZ03] S. Fomin and A. Zelevinsky. Y-systems and generalized associahedra. *Ann. of Math. (2)*, 158(3):977–1018, 2003.

[Her14] P. Hersh. Regular cell complexes in total positivity. *Invent. Math.*, 197(1):57–114, 2014.

[Kel13] B. Keller. The periodicity conjecture for pairs of Dynkin diagrams. *Ann. of Math. (2)*, 177(1):111–170, 2013.

[KS68] D. G. Kelly and S. Sherman. General Griffiths' Inequalities on Correlations in Ising Ferromagnets. *Journal of Mathematical Physics*, 9(3):466–484, 1968.

[KW41] H. A. Kramers and G. H. Wannier. Statistics of the two-dimensional ferromagnet. I. *Phys. Rev. (2)*, 60:252–262, 1941.

[KW14] Y. Kodama and L. Williams. KP solitons and total positivity for the Grassmannian. *Invent. Math.*, 198(3):637–699, 2014.

[Lus90] G. Lusztig. Canonical bases arising from quantized enveloping algebras. *J. Amer. Math. Soc.*, 3(2):447–498, 1990.

[Lus94] G. Lusztig. Total positivity in reductive groups. In *Lie theory and geometry*, volume 123 of *Progr. Math.*, pages 531–568. Birkhäuser Boston, Boston, MA, 1994.

[Lus98] G. Lusztig. Introduction to total positivity. In *Positivity in Lie theory: open problems*, volume 26 of *De Gruyter Exp. Math.*, pages 133–145. de Gruyter, Berlin, 1998.

[Pos06] A. Postnikov. Total positivity, Grassmannians, and networks. Preprint, http://math.mit.edu/~apost/papers/tpgrass.pdf, 2006.

[PSW09] A. Postnikov, D. Speyer, and L. Williams. Matching polytopes, toric geometry, and the totally non-negative Grassmannian. *J. Algebraic Combin.*, 30(2):173–191, 2009.

[RW08] K. Rietsch and L. Williams. The totally nonnegative part of G/P is a CW complex. *Transform. Groups*, 13(3-4):839–853, 2008.

[RW10] K. Rietsch and L. Williams. Discrete Morse theory for totally non-negative flag varieties. *Adv. Math.*, 223(6):1855–1884, 2010.

[Vol07] A. Y. Volkov. On the periodicity conjecture for Y-systems. *Comm. Math. Phys.*, 276(2):509–517, 2007.

[Wil07] L. K. Williams. Shelling totally nonnegative flag varieties. *J. Reine Angew. Math.*, 609:1–21, 2007.

[Zam91] A. B. Zamolodchikov. On the thermodynamic Bethe ansatz equations for reflectionless ADE scattering theories. *Phys. Lett. B*, 253(3-4):391–394, 1991.