THE TWIST FOR RICHARDSON VARIETIES

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ABSTRACT. We construct the twist automorphism of open Richardson varieties inside the flag variety of a complex semisimple algebraic group. We show that the twist map preserves totally positive parts, and prove a Chamber Ansatz formula for it. Our twist map generalizes the twist maps previously constructed by Berenstein–Fomin–Zelevinsky, Marsh–Scott, and Muller–Speyer. We use it to explain the relationship between the two conjectural cluster structures for Richardson varieties studied by Leclerc and by Ingermanson.

1. INTRODUCTION

Let G be a complex, simply-connected, semisimple algebraic group, split over $\mathbb{R}$, and let $B, B_- \subset G$ be a pair of opposite Borel subgroups. Let $H := B \cap B_-$ be the maximal torus and $W := N_G(H)/H$ be the associated Weyl group. We denote by $\ell(\cdot)$ the length function on W. Let $N \subset B$ and $N_- \subset B_-$ be the unipotent radicals. We have Bruhat decompositions

$$G = \bigsqcup_{w \in W} B_- w B_- = \bigsqcup_{v \in W} B v B_- = \bigsqcup_{v \in W} B_- v B.$$

We consider the *(generalized) flag variety* G/B_-. Our main objects of study are the *open Richardson varieties*

$$(1.1) \qquad \vec{R}^{\circ}_{v,w} := (B_- w B_- \cap B v B_-)/B_- \subset G/B_-,$$

where the intersection is nonempty whenever $v \leq w$ in the Bruhat order on W.

Our goal is to define the *twist map* $\vec{\tau}_{v,w} : \vec{R}^{\circ}_{v,w} \xrightarrow{\sim} \vec{R}^{\circ}_{v,w}$ and study its properties. Previously, twist maps have been defined for the following varieties:

(1) unipotent cells [BFZ96, BZ97];
(2) double Bruhat cells [FZ99];
(3) open positroid varieties in the Grassmannian [MaSc16, MuSp17].

Both unipotent cells and positroid varieties, as well as type A double Bruhat cells, are special cases of open Richardson varieties. We show that $\vec{\tau}_{v,w}$ specializes to the corresponding twist map in each of those cases. See also [GLS12, Wil15, Wen20, KO21] for related results on the twist map and its relation to the Donaldson–Thomas transformation.

In cases (1)–(3), the twist map has a close relationship to *total positivity* [GK02, Sch30, Lus94, Lus98] and *cluster algebras* [FZ02]. The totally positive part $X_{>0}$ of each of the above varieties X admits a *positive parametrization*, that is, a homeomorphism $\mathbf{g} : (\mathbb{R}_{>0})^{\dim X} \xrightarrow{\sim}$

$X_{>0}$. This parametrization depends on some extra combinatorial data: a reduced word in case (1), a double reduced word in case (2), and a plabic graph [Pos06] in case (3). Next, each variety X in cases (1)–(3) admits a *cluster algebra structure* [BFZ05, Sco06, GLS13, SSBW19, GL19, GY20], which in particular includes a family of regular functions on X called *cluster variables*. The twist map $\tau : X \xrightarrow{\sim} X$ was shown to be an isomorphism that *preserves total positivity*, that is, restricts to a homeomorphism $X_{>0} \xrightarrow{\sim} X_{>0}$. Moreover, it allows one to invert the positive parametrization $\mathbf{g} : (\mathbb{R}_{>0})^{\dim X} \xrightarrow{\sim} X_{>0}$ via a *Chamber Ansatz* formula: for $\mathbf{t} \in (\mathbb{R}_{>0})^{\dim X}$, the $\mathbf{t}$-parameters are given by monomials in the cluster variables evaluated at $\tau(\mathbf{g}(\mathbf{t}))$.

We show that our twist map $\vec{\tau}_{v,w}$ has analogous properties. (See Section 2 for the full description of our main results.) The totally positive part $\vec{R}_{v,w}^{>0}$ of $\vec{R}_{v,w}^{\circ}$ was defined by Lusztig [Lus94]; see also [Rie99]. Positive parametrizations of $\vec{R}_{v,w}^{>0}$ were constructed in [MR04]. They depend on the following combinatorial data: given a choice of a reduced expression $\mathbf{w}$ for w, we find a *positive distinguished subexpression* $\mathbf{v}$ for v inside $\mathbf{w}$; see Section 2.6. This gives rise to a set $J_{\mathbf{v}}^{\circ}$ of size $|J_{\mathbf{v}}^{\circ}| = \ell(w) - \ell(v) = \dim \vec{R}_{v,w}^{\circ}$, and [MR04] give a positive parametrization $\mathbf{g}_{\mathbf{v},\mathbf{w}} : (\mathbb{R}_{>0})^{J_{\mathbf{v}}^{\circ}} \xrightarrow{\sim} \vec{R}_{v,w}^{>0}$. Two conjectural cluster structures on $\vec{R}_{v,w}^{\circ}$ were studied in [Lec16, Ing19].

Theorem. *The map $\vec{\tau}_{v,w} : \vec{R}_{v,w}^{\circ} \xrightarrow{\sim} \vec{R}_{v,w}^{\circ}$ restricts to a homeomorphism $\vec{R}_{v,w}^{>0} \xrightarrow{\sim} \vec{R}_{v,w}^{>0}$ on the totally positive part of $\vec{R}_{v,w}^{\circ}$. For $\mathbf{t} \in (\mathbb{R}_{>0})^{J_{\mathbf{v}}^{\circ}}$, the $\mathbf{t}$-parameters can be recovered as monomials in the regular functions considered in* [MR04, Lec16, Ing19], *evaluated at the twist of $\mathbf{g}_{\mathbf{v},\mathbf{w}}(\mathbf{t})$.*

We note that one Chamber Ansatz formula, allowing one to recover the positive parameters of [MR04], has already been shown in [MR04]. However, we found that this was only part of the complete picture. First, recall that unipotent cells considered in [BFZ96, BZ97] are special cases of $\vec{R}_{v,w}^{\circ}$ when $v = \mathrm{id}$. It turns out that in this case, the Chamber Ansatz formula of [BZ97] is *not* a specialization of the Chamber Ansatz formula of [MR04]. The explanation for this seemingly strange phenomenon comes from the work of Muller–Speyer [MuSp17]: in the case of open positroid varieties, there are *two* Chamber Ansatz formulas: one uses the *right twist* map $\vec{\tau}$ [MuSp17], and the other one uses the *left twist* $\overleftarrow{\tau}$, which is the inverse map of $\vec{\tau}$. We show that the same situation takes place in the case of open Richardson varieties. In particular, we have two Chamber Ansatz formulas, fitting into the following commutative diagram, analogous to [MuSp17, Theorem 7.1]:

$$(1.2)$$

$$
\begin{array}{ccccc}
(\mathbb{R}_{>0})^{J_{\mathbf{v}}^{\circ}} & \xleftarrow[\text{Chamber Ansatz}]{\text{Left twist}} & (\mathbb{R}_{>0})^{J_{\mathbf{v}}^{\circ}} & \xleftarrow[\text{Chamber Ansatz}]{\text{Right twist}} & (\mathbb{R}_{>0})^{J_{\mathbf{v}}^{\circ}} \\
{\scriptstyle (\overleftarrow{f}_j)_{j \in J_{\mathbf{v}}^{\circ}}} \Big\uparrow & & {\scriptstyle \mathbf{g}_{\mathbf{v},\mathbf{w}}} \Big\downarrow & & \Big\uparrow {\scriptstyle (\vec{f}_j)_{j \in J_{\mathbf{v}}^{\circ}}} \\
\vec{R}_{v,w}^{>0} & \underset{\overleftarrow{\tau}_{v,w}}{\overset{\vec{\tau}_{v,w}}{\rightleftarrows}} & \vec{R}_{v,w}^{>0} & \underset{\overleftarrow{\tau}_{v,w}}{\overset{\vec{\tau}_{v,w}}{\rightleftarrows}} & \vec{R}_{v,w}^{>0}.
\end{array}
$$

Here, all maps are homeomorphisms. The regular functions $\vec{f}_j$ are products of cluster variables considered in [Lec16], while the regular functions $\overleftarrow{f}_j$ are the *chamber minors* considered in [MR04]. (They are also closely related to the products of cluster variables considered in [Ing19].) The Chamber Ansatz of [BZ97] corresponds to the right twist, while the Chamber Ansatz of [MR04] corresponds to the left twist. We have a complex-algebraic version of

the above diagram:

$$(1.3)$$

$$
\begin{array}{ccccc}
\mathbb{C}^{J_{\mathbf{v}}^{\circ}} & \xleftarrow{\quad \text{Left twist} \quad}_{\text{Chamber Ansatz}} & (\mathbb{C}^{*})^{J_{\mathbf{v}}^{\circ}} & \xleftarrow{\quad \text{Right twist} \quad}_{\text{Chamber Ansatz}} & \mathbb{C}^{J_{\mathbf{v}}^{\circ}} \\
\big\uparrow {\scriptstyle (\bar{f}_j)_{j \in J_{\mathbf{v}}^{\circ}}} & & \big\downarrow {\scriptstyle \mathbf{g}_{\mathbf{v},\mathbf{w}}} & & \big\uparrow {\scriptstyle (\vec{f}_j)_{j \in J_{\mathbf{v}}^{\circ}}} \\
\overleftarrow{R}_{v,w}^{\circ} & \underset{\overleftarrow{\tau}_{v,w}}{\overset{\vec{\tau}_{v,w}}{\rightleftarrows}} & \vec{R}_{v,w}^{\circ} & \underset{\overleftarrow{\tau}_{v,w}}{\overset{\vec{\tau}_{v,w}}{\rightleftarrows}} & \overrightarrow{R}_{v,w}^{\circ}.
\end{array}
$$

Here, the solid arrows denote regular maps and the horizontal dashed arrows denote rational monomial maps.

The original motivation for our work was to understand the relationship between the conjectural cluster structures on $\vec{R}_{v,w}^{\circ}$ considered in [Lec16, Ing19]. In Section 11, we use the map $\vec{\tau}_{v,w}$ to give a precise comparison.

We also consider the left-sided quotient $B_{-} \backslash G$. Let

$$(1.4) \qquad \overleftarrow{R}_{v,w}^{\circ} := B_{-} \backslash (B_{-} w B_{-} \cap B_{-} v B) \subset B_{-} \backslash G.$$

The intersection is again nonempty whenever $v \leq w$. One of the important ingredients of our construction is the *chiral map*

$$(1.5) \qquad \vec{\chi}_{v} : \overleftarrow{R}_{v,w}^{\circ} \xrightarrow{\sim} \vec{R}_{v,w}^{\circ},$$

obtained by identifying both sides with $N\dot{v} \cap \dot{v}N \cap B_{-}wB_{-}$, where N is the unipotent radical of B and $\dot{v} \in N_{G}(H)$ is a lift of $v \in W$ to G. We show that $\vec{\chi}_{v}$ preserves total positivity, i.e., restricts to a homeomorphism $\overleftarrow{R}_{v,w}^{>0} \xrightarrow{\sim} \vec{R}_{v,w}^{>0}$. The significance of the chiral map first occurred to us during the development of the results of [GKL21]. Later, this map played a key role in [GL19]. In this paper, we use it in the construction of the twist map: by definition, $\vec{\tau}_{v,w}$ is the composition of $\vec{\chi}_{v}$ with the *right pre-twist map* $\vec{\tau}_{v,w}^{\mathrm{pre}} : \vec{R}_{v,w}^{>0} \xrightarrow{\sim} \overleftarrow{R}_{v,w}^{>0}$.

Outline. In Section 2, we discuss our main results in detail. We review background material in Section 3. In Section 4, we show that the pre-twist and twist maps are well defined; cf. Proposition 2.4. In Section 5, we review the Chamber Ansatz of [MR04] and use it to deduce the left twist Chamber Ansatz (Theorem 2.12). We show that the chiral map — as well as some other related "reversal" maps — preserves total positivity (Theorem 2.3) in Section 6. In Section 7, we prove the analogous statement for the pre-twist and the twist maps (Theorem 2.6). We prove the right twist Chamber Ansatz (Theorem 2.8) in Section 8. In Section 9, we show that our twist map specializes to the twist of [MuSp17] in the case of open positroid varieties (Theorem 2.7). We show that the chiral and the (pre-)twist maps are given by subtraction-free rational functions in Section 10. In Section 11, we explain how our results connect the conjectural cluster structures on $\vec{R}_{v,w}^{\circ}$ studied in [Lec16, Ing19]. Finally, in Appendix A, we discuss fixed points of the twist map.

2. Main results

We explain our main results in detail. The reader is encouraged to refer to Table 1 for the notation and statements used throughout the paper, and to Section 2.9 for an example.

2.1. **Preliminaries.** We fix a *pinning* of G, which in particular includes a choice $\{\alpha_i\}_{i \in I}$ of simple roots for the root system Φ of G, and a homomorphism $\phi_i : \mathrm{SL}_2(\mathbb{C}) \to G$ for each $i \in I$. We set

$$x_i(t) := \phi_i \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}, \quad y_i(t) := \phi_i \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}, \quad \text{and} \quad \dot{s}_i := \phi_i \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = x_i(1)y_i(-1)x_i(1).$$

We have an involutive anti-automorphism $g \mapsto g^T$ of G defined by $x_i(t)^T = y_i(t)^T$, $y_i(t)^T = x_i(t)^T$, and $a^T = a$ for all $a \in H$.

Lusztig [Lus94] defined the *totally nonnegative part* $G_{\geq 0}$ of G as the submonoid generated by $x_i(t), y_i(t)$ for $t > 0$, as well as by the totally positive elements of H; see Section 3.1 for a precise definition. He defined the *totally nonnegative flag variety* $(G/B_-)_{\geq 0}$ to be the closure of the image of $G_{\geq 0}$ under the projection $G \to G/B_-$. Similarly, define $(B_- \backslash G)_{\geq 0} \subset B_- \backslash G$ to be the closure of the image of $G_{\geq 0}$ under the projection $G \to B_- \backslash G$. Letting $\vec{R}^{>0}_{v,w} := \vec{R}^{\circ}_{v,w} \cap (G/B_-)_{\geq 0}$ and $\reflectbox{$\vec{R}$}^{>0}_{v,w} := \reflectbox{$\vec{R}$}^{\circ}_{v,w} \cap (B_- \backslash G)_{\geq 0}$, we have stratifications

$$G/B_- = \bigsqcup_{v \leq w} \vec{R}^{\circ}_{v,w}, \quad B_- \backslash G = \bigsqcup_{v \leq w} \reflectbox{$\vec{R}$}^{\circ}_{v,w}, \quad (G/B_-)_{\geq 0} = \bigsqcup_{v \leq w} \vec{R}^{>0}_{v,w}, \quad (B_- \backslash G)_{\geq 0} = \bigsqcup_{v \leq w} \reflectbox{$\vec{R}$}^{>0}_{v,w}.$$

For each element $w \in W$, a *reduced word* for w is an expression of the form $w = s_{i_1} s_{i_2} \cdots s_{i_m}$, where m is minimal possible. We refer to such m as the *length* of w and denote $\ell(w) := m$. Given such a reduced word, we set $\dot{w} := \dot{s}_{i_1} \dot{s}_{i_2} \cdots \dot{s}_{i_m}$. The resulting element $\dot{w} \in G$ does not depend on the choice of the reduced word. We omit the dot when the choice of the representative of w inside $N_G(H)$ does not matter; for example, we write $B_- w B_-$ instead of $B_- \dot{w} B_-$.

2.2. **Type A_{n-1}.** Let $G = \mathrm{SL}_n(\mathbb{C})$, in which case B, B_-, N, N_-, and H are the subgroups consisting of upper triangular, lower triangular, upper unitriangular, lower unitriangular, and diagonal matrices, respectively. The Weyl group W is the symmetric group S_n. For $j \in [n] := \{1, 2, \dots, n\}$ and a permutation $w = s_{i_1} \cdots s_{i_m}$, we set $w(j) := s_{i_1}(\cdots (s_{i_m}(j)) \cdots)$.

The submonoid $G_{\geq 0}$ consists of matrices all of whose minors (of all sizes) are nonnegative. For two sets $I, J \subset [n]$ of the same size k, let Δ^I_J denote the $k \times k$ minor with row set I and column set J. We say that Δ^I_J is a *right-aligned flag minor* if $J = \{n - k + 1, \dots, n - 1, n\}$ consists of the rightmost k columns. Similarly, Δ^I_J is a *top-aligned flag minor* if $I = [k]$ consists of the top k rows.

For a matrix $g \in G$, the flag gB_- belongs to $(G/B_-)_{\geq 0}$ if and only if the right-aligned flag minors of g are all nonnegative. (More precisely, if and only if for each $k \in [n]$, the ratio of any two $k \times k$ nonzero right-aligned flag minors of g is nonnegative.) Similarly, $B_- g$ belongs to $(B_- \backslash G)_{\geq 0}$ if and only if the top-aligned flag minors of g are all nonnegative. For a permutation $v \in W$, the matrix $\dot{v} \in G$ is a signed permutation matrix for v, where the signs are chosen so that the right-aligned (equivalently, the top-aligned) flag minors of $\dot{v}$ are nonnegative. For example, if $v = s_1 s_2 = [2, 3, 1, 4]$ in one-line notation then

$$(2.1) \qquad\qquad \dot{v} = \dot{s}_1 \dot{s}_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

2.3. **The chiral map.** Let $v \leq w$ in W. Our first main result is a construction of a natural map relating $\reflectbox{$\vec{R}$}^{\circ}_{v,w} \subset B_- \backslash G$ and $\vec{R}^{\circ}_{v,w} \subset G/B_-$.

Notation 2.1. For a subset $F \subset G$, we set $F^{\cap w} := F \cap B_- w B_-$.

It is known [BGY06, Lec16] that the maps $B_-g \hookleftarrow g \mapsto gB_-$ restrict to isomorphisms

$$(2.2) \qquad \overleftarrow{R}^\circ_{v,w} \xleftarrow{\sim} (N\dot{v} \cap \dot{v}N)^{\cap w} \xrightarrow{\sim} \overrightarrow{R}^\circ_{v,w}.$$

Definition 2.2. The *chiral map* $\vec{\chi}_v : \overleftarrow{R}^\circ_{v,w} \xrightarrow{\sim} \overrightarrow{R}^\circ_{v,w}$ is obtained by identifying both spaces with $(N\dot{v} \cap \dot{v}N)^{\cap w}$ via (2.2). The inverse of $\vec{\chi}_v$ is denoted $\overleftarrow{\chi}_v$.

Taking the union over all $w \in W$ satisfying $w \geq v$, we get an isomorphism

$$\vec{\chi}_v : B_- \backslash (B_- v B) \xrightarrow{\sim} (BvB_-)/B_-$$

on the *opposite Schubert cells* obtained by identifying both spaces with $N\dot{v} \cap \dot{v}N$. We state our first result: the chiral map preserves total positivity.

Theorem 2.3. *Let $v \in W$. Then for any $g \in N\dot{v} \cap \dot{v}N$, we have*

$$(2.3) \qquad gB_- \in (G/B_-)_{\geq 0} \iff B_-g \in (B_- \backslash G)_{\geq 0}.$$

In other words, for all $v \leq w \in W$, the map $\vec{\chi}_v : \overleftarrow{R}^\circ_{v,w} \xrightarrow{\sim} \overrightarrow{R}^\circ_{v,w}$ restricts to a homeomorphism

$$\vec{\chi}_v : \overleftarrow{R}^{>0}_{v,w} \xrightarrow{\sim} \overrightarrow{R}^{>0}_{v,w}.$$

Consider the case $G = \mathrm{SL}_n(\mathbb{C})$. Then the set $N\dot{v} \cap \dot{v}N$ consists of matrices obtained from the signed permutation matrix $\dot{v}$ by allowing nonzero entries in the cells which are both to the right and above a ± 1 in $\dot{v}$. E.g., for $v = [2, 3, 1, 4]$ and $\dot{v}$ as in (2.1), the set $N\dot{v} \cap \dot{v}N$ consists of matrices

$$(2.4) \qquad \begin{pmatrix} 0 & 0 & 1 & a \\ -1 & -d & 0 & b \\ 0 & -1 & 0 & c \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{for } a, b, c, d \in \mathbb{C}.$$

Then Theorem 2.3 claims that a matrix $g \in N\dot{v} \cap \dot{v}N$ has all right-aligned flag minors nonnegative if and only if it has all top-aligned flag minors nonnegative. We encourage the reader to check that either one of these two conditions is satisfied for the matrix in (2.4) whenever $a, b, c, d \in \mathbb{R}$ satisfy

$$(2.5) \qquad a, b, c, d, cd - b \geq 0.$$

2.4. The twist map. Let $G_0^{\pm} := BB_-$ and $G_0^{\mp} := B_-B$. It is well known that the multiplication map gives isomorphisms

$$(2.6) \qquad N \times H \times N_- \xrightarrow{\sim} G_0^{\pm} \quad \text{and} \quad N_- \times H \times N \xrightarrow{\sim} G_0^{\mp}.$$

We denote the inverses of these maps by $g \mapsto ([g]_L^+, [g]_0^{\pm}, [g]_R^-)$ and $g \mapsto ([g]_L^-, [g]_0^{\mp}, [g]_R^+)$, respectively. Here $[g]_L^+, [g]_R^+ \in N$, $[g]_R^-, [g]_L^- \in N_-$, and $[g]_0^{\pm}, [g]_0^{\mp} \in H$. In type A, the factorizations

$$(2.7) \qquad g = [g]_L^+ \cdot [g]_0^{\pm} \cdot [g]_R^- \quad \text{and} \quad g = [g]_L^- \cdot [g]_0^{\mp} \cdot [g]_R^+$$

are known as the *UDL and LDU decompositions*, respectively. We will be mostly interested in the elements $[g]_L^+$ and $[g]_R^+$.

The twist map essentially boils down to the following isomorphism.

Proposition 2.4. *Let $v \leq w$ in W. We have an isomorphism*

$$(N\dot{v})^{\cap w} \xrightarrow{\sim} (\dot{v}N)^{\cap w}, \quad g \mapsto \dot{v}[g^T \dot{w}]_L^+.$$

The inverse is given by $h \mapsto [\dot{w}h^T]_R^+ \dot{v}$.

It turns out that the map of Proposition 2.4 descends to a map $\vec{R}^\circ_{v,w} \xrightarrow{\sim} \overleftarrow{R}^\circ_{v,w}$. We refer to the resulting map as the *right pre-twist*.

Definition 2.5. Define the *right pre-twist* $\vec{\tau}^{\mathrm{pre}}_{v,w}$ and the *left pre-twist* $\overleftarrow{\tau}^{\mathrm{pre}}_{v,w}$ by

$$\vec{\tau}^{\mathrm{pre}}_{v,w} : \vec{R}^\circ_{v,w} \xrightarrow{\sim} \overleftarrow{R}^\circ_{v,w}, \quad gB_- \mapsto B_- v[g^T \dot{w}]^+_L \quad \text{for } g \in (N\dot{v})^{\cap w};$$

$$\overleftarrow{\tau}^{\mathrm{pre}}_{v,w} : \overleftarrow{R}^\circ_{v,w} \xrightarrow{\sim} \vec{R}^\circ_{v,w}, \quad B_- h \mapsto [\dot{w}h^T]^+_R v B_- \quad \text{for } h \in (\dot{v}N)^{\cap w}.$$

The *right twist* $\vec{\tau}_{v,w}$ and the *left twist* $\overleftarrow{\tau}_{v,w}$ are defined by composing the pre-twists with the chiral map:

$$\vec{\tau}_{v,w} := \vec{\chi}_v \circ \vec{\tau}^{\mathrm{pre}}_{v,w} : \vec{R}^\circ_{v,w} \xrightarrow{\sim} \vec{R}^\circ_{v,w}, \qquad \overleftarrow{\tau}_{v,w} := \overleftarrow{\tau}^{\mathrm{pre}}_{v,w} \circ \overleftarrow{\chi}_v : \vec{R}^\circ_{v,w} \xrightarrow{\sim} \vec{R}^\circ_{v,w}.$$

It follows from Proposition 2.4 that the maps $\vec{\tau}_{v,w}$ and $\overleftarrow{\tau}_{v,w}$ are mutually inverse automorphisms of $\vec{R}^\circ_{v,w}$. It turns out that these maps also preserve total positivity.

Theorem 2.6. *For all $v \leq w$ in W, the twist map restricts to a homeomorphism*

$$\vec{\tau}_{v,w} : \vec{R}^{>0}_{v,w} \xrightarrow{\sim} \vec{R}^{>0}_{v,w}.$$

See Section 2.9 for an example.

In fact, the twist map and the chiral map have the property of being given by *subtraction-free rational expressions*; see Section 10.

2.5. **Muller–Speyer twist.** Let $G = \mathrm{SL}_n(\mathbb{C})$, and thus $I = [n-1]$. Choose $k \in I$ and set $J := I \setminus \{k\}$. Let $P^J_- \subset G$ be the parabolic subgroup consisting of block lower triangular matrices of the form $\begin{pmatrix} A & 0 \\ C & D \end{pmatrix}$, where $A \in \mathrm{GL}_k(\mathbb{C})$ and $D \in \mathrm{GL}_{n-k}(\mathbb{C})$. The quotient G/P^J_- is isomorphic to the Grassmannian $\mathrm{Gr}(n-k, n)$ of $(n-k)$-planes inside $\mathbb{C}^n$ via the map sending $gP^J_- \in G/P^J_-$ to the column span of the rightmost $n-k$ columns of g. We say that $w \in S_n$ is *k-Grassmannian* if $w(1) < \cdots < w(k)$ and $w(k+1) < \cdots < w(n)$, and we let W^J denote the set of k-Grassmannian permutations. Let $Q^J := \{(v, w) \in W \times W^J \mid v \leq w\}$. Then the elements of Q^J label *open positroid varieties* [KLS13] inside $\mathrm{Gr}(n-k, n)$. Specifically, for each $(v, w) \in Q^J$, the natural projection map $\pi_J : G/B_- \to G/P^J_-$ restricts to an isomorphism $\vec{R}^\circ_{v,w} \xrightarrow{\sim} \Pi^\circ_{v,w}$, where we set $\Pi^\circ_{v,w} := \pi_J(\vec{R}^\circ_{v,w})$. We have a stratification

$$\mathrm{Gr}(n-k, n) = \bigsqcup_{(v,w) \in Q^J} \Pi^\circ_{v,w}.$$

For each $(v, w) \in Q^J$, Muller and Speyer [MuSp17] defined the *right* and *left twist* automorphisms of $\Pi^\circ_{v,w}$. Even though their definition is quite different from ours, we show that their twist maps are special cases of our twist maps.

Theorem 2.7. *For all $(v, w) \in Q^J$, under the identification $\pi_J : \vec{R}^\circ_{v,w} \xrightarrow{\sim} \Pi^\circ_{v,w}$, the maps $\vec{\tau}_{v,w}$ and $\overleftarrow{\tau}_{v,w}$ coincide respectively with the right and left twist maps defined in* [MuSp17].

See Example 9.2. In particular, up to the action of H, our twist maps generalize the twist maps of Marsh–Scott [MaSc16]. In view of [MuSp17, Section A.4], we also obtain the twist map of [FZ99] for type A double Bruhat cells as a special case.

2.6. **MR-parametrizations.** Fix $v \leq w$ in W. Let $w = s_{i_1} s_{i_2} \cdots s_{i_m}$ be a reduced word for $w \in W$. In this case, $\mathbf{w} := (i_1, \ldots, i_m)$ is called a *reduced expression* for w. By [MR04, Lemma 3.5], every reduced expression $\mathbf{w}$ for w contains a unique "rightmost" reduced subexpression $\mathbf{v}$ for v, called the *positive distinguished subexpression*. We denote the set of such pairs $(\mathbf{v}, \mathbf{w})$ by $\mathrm{Red}(v, w)$, and for $(\mathbf{v}, \mathbf{w}) \in \mathrm{Red}(v, w)$, we let $J_{\mathbf{v}}^{\circ} \subset [m]$ denote the set of indices *not* used in $\mathbf{v}$.

The space $\vec{R}_{v,w}^{\circ}$ contains an open dense subset, called the *open Deodhar stratum*. It admits the following *Marsh–Rietsch (MR) parametrization*, introduced in [MR04]. For $\mathbf{t} = (t_r)_{r \in J_{\mathbf{v}}^{\circ}} \in (\mathbb{C}^*)^{|J_{\mathbf{v}}^{\circ}|}$, define an element

$$(2.8) \qquad \mathbf{g}_{\mathbf{v},\mathbf{w}}(\mathbf{t}) = g_1 \cdots g_m \in (N\dot{v})^{\cap w}, \qquad \text{where} \qquad g_r = \begin{cases} \dot{s}_{i_r} & \text{if } r \notin J_{\mathbf{v}}^{\circ}, \\ x_{i_r}(t_r) & \text{if } r \in J_{\mathbf{v}}^{\circ}. \end{cases}$$

The map $(\mathbb{C}^*)^{J_{\mathbf{v}}^{\circ}} \to \vec{R}_{v,w}^{\circ}$ given by $\mathbf{t} \mapsto \mathbf{g}_{\mathbf{v},\mathbf{w}}(\mathbf{t}) B_-$ is an isomorphism onto its image, the open Deodhar stratum in $\vec{R}_{v,w}^{\circ}$. The restriction of this map to $(\mathbb{R}_{>0})^{J_{\mathbf{v}}^{\circ}}$ gives a homeomorphism $(\mathbb{R}_{>0})^{J_{\mathbf{v}}^{\circ}} \xrightarrow{\sim} \vec{R}_{v,w}^{>0}$.

2.7. **Chamber Ansatz.** For $r \in [m]$, we set

$$w_{(r)} := s_{i_1} \cdots s_{i_r}, \qquad v_{(r)} := s_{i_1}^{\mathbf{v}} \cdots s_{i_r}^{\mathbf{v}}, \quad \text{where} \quad s_{i_r}^{\mathbf{v}} := \begin{cases} s_{i_r}, & \text{if } r \notin J_{\mathbf{v}}^{\circ}, \\ 1, & \text{if } r \in J_{\mathbf{v}}^{\circ}; \end{cases}$$

$$w^{(r)} := w^{-1} \cdot w_{(r)} = s_{i_m} \cdots s_{i_{r+1}}, \qquad \text{and} \qquad v^{(r)} := v^{-1} \cdot v_{(r)} = s_{i_m}^{\mathbf{v}} \cdots s_{i_{r+1}}^{\mathbf{v}}.$$

Let $\{\omega_i\}_{i \in I}$ be the fundamental weights corresponding to the simple roots $\{\alpha_i\}_{i \in I}$. For each $i \in I$ and $v, w \in W$, we have a regular function $\Delta_{w\omega_i}^{v\omega_i} : G \to \mathbb{C}$, called a *generalized minor* [FZ99]. The function $\Delta_{w\omega_i}^{v\omega_i}$ depends only on the weights $v\omega_i, w\omega_i$. In the case $G = \mathrm{SL}_n(\mathbb{C})$, $\Delta_{w\omega_i}^{v\omega_i}$ is the usual matrix minor with row set $\{v(1), \ldots, v(i)\}$ and column set $\{w(1), \ldots, w(i)\}$.

For $r \in [m]$, set

$$(2.9) \qquad \vec{f}_r := \Delta_{w^{(r-1)}\omega_{i_r}}^{v^{(r-1)}\omega_{i_r}} : N \to \mathbb{C}.$$

Leclerc [Lec16] shows that the irreducible factors of the generalized minors $\vec{f}_1, \vec{f}_2, \ldots, \vec{f}_m$ form a seed of a cluster subalgebra $\vec{\mathcal{A}} \subset \mathbb{C}[\vec{R}_{v,w}^{\circ}]$ of the coordinate ring of $\vec{R}_{v,w}^{\circ}$.

Let $(a_{i,j})_{i,j \in I}$ denote the Cartan matrix of the root system Φ of G, given by $a_{i,j} := \langle \alpha_i, \alpha_j^{\vee} \rangle$; see Section 3.1 for further details. We prove the following result.

Theorem 2.8 (Right twist Chamber Ansatz). *Let $v \leq w$ in W and $(\mathbf{v}, \mathbf{w}) \in \mathrm{Red}(v, w)$. Let $g := \mathbf{g}_{\mathbf{v},\mathbf{w}}(\mathbf{t})$ be MR-parametrized, and let $\vec{y} := [g^T \dot{w}]_L^+$ (cf. Proposition 2.4). Then for all $r \in J_{\mathbf{v}}^{\circ}$, we have*

$$(2.10) \qquad t_r = \frac{\prod_{j \neq i_r} \Delta_{w^{(r)}\omega_j}^{v^{(r)}\omega_j}(\vec{y})^{-a_{j,i_r}}}{\Delta_{w^{(r)}\omega_{i_r}}^{v^{(r)}\omega_{i_r}}(\vec{y}) \Delta_{w^{(r-1)}\omega_{i_r}}^{v^{(r-1)}\omega_{i_r}}(\vec{y})}.$$
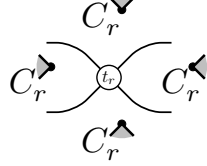
Conversely, for each $j \in [m]$, we have

$$(2.11) \qquad \vec{f}_j(\vec{y}) = \prod_{r \in J_{\mathbf{v}}^{\circ} : r \geq j} t_r^{-\langle \omega_{i_j}, s_{i_j} s_{i_{j+1}} \cdots s_{i_{r-1}} \alpha_{i_r}^{\vee} \rangle}.$$

Remark 2.9. Each non-trivial (i.e., not identically equal to 1) generalized minor of $\vec{y}$ appearing in the right-hand side of (2.10) is of the form $\vec{f}_j(\vec{y})$ for some $j \in [m]$. As we explain in Section 11.2, $\vec{f}_j(\vec{y})$ is a function of the right twist $\vec{\tau}_{v,w}(gB_-)$. In particular, (2.10) gives a way to recover the MR-parameters $(t_r)_{r \in J_{\mathbf{v}}^{\circ}}$ from $\vec{\tau}_{v,w}(gB_-)$. See also [BH21, Equation (4-1)].

Remark 2.10. Let $v = \mathrm{id}$. Then $\vec{R}_{v,w}^{\circ} \cong N^{\cap w}$ is a unipotent cell, considered in [BFZ96, BZ97]. It is clear that the twist map $\eta_w : N^{\cap w} \xrightarrow{\sim} N^{\cap w}$ defined in [BZ97, Theorem 1.2] coincides with the map $h \mapsto [\dot{w}h^T]_R^+$ of Proposition 2.4. The Chamber Ansatz in [BZ97, Theorem 1.4] describes the $\mathbf{t}$-parameters in terms of the functions $\vec{f}_j$ evaluated at the element $\eta_w^{-1}(g) = \vec{y}$. Thus [BZ97, Theorem 1.4] is a special case of Theorem 2.8.

Remark 2.11. Even though (2.10) looks almost identical to [MR04, Theorem 7.1(1)], the two Chamber Ansatz formulas are genuinely different, and (2.10) cannot be deduced from the results of [MR04] in a simple way. The crucial difference between (2.10) and [MR04, Theorem 7.1(1)] is that our formula involves $v^{(r)}, w^{(r)}$ while [MR04, Theorem 7.1(1)] involves $v_{(r)}, w_{(r)}$ instead. See Figure 1 for a comparison.

In order to complete the diagrams in (1.2)–(1.3), we need to state the *left twist Chamber Ansatz*, which is much closer to the results of [MR04]. Note that we work inside G/B_- while [MR04] work inside G/B. To relate our results with those in [MR04], we apply the involution $x \mapsto x^{\theta}$ of [FZ02, Eq. (1.11)]; see Remark 3.1.

Let $w_0 \in W$ be the longest element of W. For each $i \in I$, let $i^* \in I$ be defined by $\omega_{i^*} = -w_0\omega_i$. Set

$$(2.12) \qquad \overleftarrow{f}_r := \Delta_{v_{(r)}w_0\omega_{i_r^*}}^{w_{(r)}w_0\omega_{i_r^*}} : N \to \mathbb{C}.$$

Theorem 2.12 (Left twist Chamber Ansatz [MR04])**.** *Let $v \leq w$ in W and $(\mathbf{v}, \mathbf{w}) \in \mathrm{Red}(v, w)$. Let $g := \mathbf{g}_{\mathbf{v},\mathbf{w}}(\mathbf{t}) \in (N\dot{v})^{\cap w}$ be MR-parametrized, and let $g_{\cap} \in (N\dot{v} \cap \dot{v}N)^{\cap w}$ be the unique element satisfying $gB_- = g_{\cap}B_-$. Let $\overleftarrow{y} := [\dot{w}g_{\cap}^T]_R^+$ (cf. Proposition 2.4). Then for all $r \in J_{\mathbf{v}}^{\circ}$, we have*

$$(2.13) \qquad t_r = \frac{\prod_{j \neq i_r^*} \Delta_{v_{(r)}w_0\omega_j}^{w_{(r)}w_0\omega_j}(\overleftarrow{y})^{-a_{j,i_r^*}}}{\Delta_{v_{(r)}w_0\omega_{i_r^*}}^{w_{(r)}w_0\omega_{i_r^*}}(\overleftarrow{y})\Delta_{v_{(r-1)}w_0\omega_{i_r^*}}^{w_{(r-1)}w_0\omega_{i_r^*}}(\overleftarrow{y})}.$$

Conversely, for each $j \in [m]$, we have

$$(2.14) \qquad \overleftarrow{f}_j(\overleftarrow{y}) = \prod_{r \in J_{\mathbf{v}}^{\circ}:\, r \leq j} t_r^{-\langle \omega_{i_j}, s_{i_j}s_{i_{j-1}}\cdots s_{i_{r+1}}\alpha_{i_r}^{\vee}\rangle}.$$

Similarly to Remark 2.9, all non-trivial generalized minors in the right-hand side of (2.13) are of the form $\overleftarrow{f}_j(\overleftarrow{y})$ for some $j \in [m]$.

Remark 2.13. After switching from G/B_- to G/B, the left twist Chamber Ansatz (2.13) coincides with [MR04, Theorem 7.1(1)], except that the element $\overleftarrow{y}$ is replaced with the transpose z^T of an arbitrary element $z \in N_-$ such that $zwB_- = gB_-$; see Theorem 5.1. We show in Section 5 that the element $z := \overleftarrow{y}^T$ satisfies these assumptions, and thus deduce (2.13) from the results of [MR04].

2.8. Type A Chamber Ansatz. When $G = \mathrm{SL}_n(\mathbb{C})$, our results admit a pictorial interpretation in terms of wiring diagrams. Fix $v \leq w$ in W and $(\mathbf{v}, \mathbf{w}) \in \mathrm{Red}(v, w)$. We construct the *wiring diagram* of $\mathbf{w} = (i_1, i_2, \ldots, i_m)$ by drawing n horizontal strands and placing, for each $r \in [m]$, a crossing with horizontal coordinate r between the strands with vertical coordinates i_r and $i_r + 1$. We refer to i_r as the *height* of this crossing. For each $r \in J_{\mathbf{v}}^\circ$, we place a dot labeled t_r on the corresponding crossing. The connected components of the complement of the wiring diagram are called *chambers*. The *height* of a chamber is the number of strands passing below it. For $r \in [m]$, we denote by $C_r^{\blacktriangleleft}, C_r^{\blacktriangledown}, C_r^{\blacktriangleright}, C_r^{\blacktriangle}$ the chambers located immediately to the right, above, to the left, and below the r-th crossing, respectively:

$$
\begin{array}{c}
C_r^{\blacktriangledown} \\
C_r^{\blacktriangleright} \quad \overset{t_r}{\times} \quad C_r^{\blacktriangleleft} \\
C_r^{\blacktriangle}
\end{array}
$$

We refer to the strands in the wiring diagram of $\mathbf{w}$ as $\mathbf{w}$-*strands*. A $\mathbf{v}$-*strand* is a path from left to right that follows the crossings not in $J_{\mathbf{v}}^\circ$ and ignores the crossings in $J_{\mathbf{v}}^\circ$. The right and left endpoints of the strands are labeled $1, 2, \ldots, n$ from the bottom to the top. For a chamber C, we define subsets $I_{\rightarrow}^{\mathbf{v}}(C), I_{\rightarrow}^{\mathbf{w}}(C), I_{\mathbf{v}}^{\leftarrow}(C), I_{\mathbf{w}}^{\leftarrow}(C) \subset [n]$ by

$$I_{\rightarrow}^{\mathbf{v}}(C) := \{\text{right endpoints of } \mathbf{v}\text{-strands passing below } C\};$$
$$I_{\rightarrow}^{\mathbf{w}}(C) := \{\text{right endpoints of } \mathbf{w}\text{-strands passing below } C\};$$
$$I_{\mathbf{v}}^{\leftarrow}(C) := \{\text{left endpoints of } \mathbf{v}\text{-strands passing above } C\};$$
$$I_{\mathbf{w}}^{\leftarrow}(C) := \{\text{left endpoints of } \mathbf{w}\text{-strands passing above } C\}.$$

Finally, we introduce *chamber minors*

$$\vec{\Delta}_C := \Delta_{I_{\rightarrow}^{\mathbf{w}}(C)}^{I_{\rightarrow}^{\mathbf{v}}(C)} \quad \text{and} \quad \overleftarrow{\Delta}_C := \Delta_{I_{\mathbf{v}}^{\leftarrow}(C)}^{I_{\mathbf{w}}^{\leftarrow}(C)}.$$

Comparing to (2.9) and (2.12), we see that for all $r \in [m]$, we have

(2.15)
$$\vec{f}_r = \vec{\Delta}_{C_r^{\blacktriangleright}} \quad \text{and} \quad \overleftarrow{f}_r = \overleftarrow{\Delta}_{C_r^{\blacktriangleleft}}.$$

Remark 2.14. We emphasize that there is no symmetry between $\vec{\Delta}_C$ and $\overleftarrow{\Delta}_C$ since both of them are defined in terms of the *rightmost* subexpression for v inside $\mathbf{w}$.

Let $r \in [m]$ and consider the two $\mathbf{w}$-strands S_1, S_2 participating in the r-th crossing. Let $\wedge_r^{\blacktriangleright}$ (resp., $\wedge_r^{\blacktriangleleft}$) denote the union of chambers located vertically between S_1 and S_2 to the left (resp., to the right) of the r-th crossing. Following [MuSp17], we refer to $\wedge_r^{\blacktriangleright}$ and $\wedge_r^{\blacktriangleleft}$ as the *upstream wedge* and the *downstream wedge*, respectively. The results of the previous subsection can be specialized as follows.

Theorem 2.15 (Type A Chamber Ansatz). *Let $G = \mathrm{SL}_n(\mathbb{C})$, $v \leq w$ in W, and $(\mathbf{v}, \mathbf{w}) \in \mathrm{Red}(v, w)$. Let $g := \mathbf{g}_{\mathbf{v},\mathbf{w}}(\mathbf{t})$ and $\vec{y} := [g^T \dot{w}]_L^+$ be as in Theorem 2.8. Let $g_\cap \in N\dot{v} \cap \dot{v}N$ with $gB_- = g_\cap B_-$ and $\overleftarrow{y} := [\dot{w} g_\cap^T]_R^+$ be as in Theorem 2.12. Then for all $r \in J_{\mathbf{v}}^\circ$,*

(2.16)
$$t_r = \frac{\vec{\Delta}_{C_r^{\blacktriangledown}}(\vec{y}) \vec{\Delta}_{C_r^{\blacktriangle}}(\vec{y})}{\vec{\Delta}_{C_r^{\blacktriangleright}}(\vec{y}) \vec{\Delta}_{C_r^{\blacktriangleleft}}(\vec{y})} = \frac{\overleftarrow{\Delta}_{C_r^{\blacktriangledown}}(\overleftarrow{y}) \overleftarrow{\Delta}_{C_r^{\blacktriangle}}(\overleftarrow{y})}{\overleftarrow{\Delta}_{C_r^{\blacktriangleright}}(\overleftarrow{y}) \overleftarrow{\Delta}_{C_r^{\blacktriangleleft}}(\overleftarrow{y})}.$$

Conversely, for each chamber C of the wiring diagram of $\mathbf{w}$, we have

$$\vec{\Delta}_C(\vec{y}) = \prod_{r \in J_{\mathbf{v}}^\circ : C \in \wedge_r^{\rotatebox{-20}{$\blacktriangleright$}}} t_r^{-1} \quad and \quad \overleftarrow{\Delta}_C(\overleftarrow{y}) = \prod_{r \in J_{\mathbf{v}}^\circ : C \in \wedge_r^{\rotatebox{20}{$\blacktriangleleft$}}} t_r^{-1}.$$

2.9. **Type A example.** Let $G = \mathrm{SL}_n(\mathbb{C})$ for $n = 4$, and let $v = s_1 s_2$ as in (2.1). Let $w = w_0$, and let $\mathbf{w} = (2, 1, 2, 3, 2, 1)$ be a reduced expression for w. We have

$$g = \mathbf{g}_{\mathbf{v},\mathbf{w}}(\mathbf{t}) = x_2(t_1)\dot{s}_1 x_2(t_3) x_3(t_4)\dot{s}_2 x_1(t_6) = \begin{pmatrix} 0 & -t_3 & 1 & t_3 t_4 \\ -1 & -t_1 - t_6 & 0 & t_1 t_4 \\ 0 & -1 & 0 & t_4 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad g_\cap = \begin{pmatrix} 0 & 0 & 1 & t_3 t_4 \\ -1 & -t_1 - t_6 & 0 & t_1 t_4 \\ 0 & -1 & 0 & t_4 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

where we assume $t_r > 0$ for all $r \in J_{\mathbf{v}}^\circ = \{1, 3, 4, 6\}$. In terms of the (a, b, c, d)-parameters in (2.4), we have $a = t_3 t_4$, $b = t_1 t_4$, $c = t_4$, and $d = t_1 + t_6$. Computing the UDL decomposition of $g^T \dot{w}$, we get

$$(2.17) \quad g^T \dot{w} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & -1 & t_1 + t_6 & -t_3 \\ 0 & 0 & 0 & 1 \\ -1 & t_4 & -t_1 t_4 & t_3 t_4 \end{pmatrix} = \begin{pmatrix} 1 & \frac{1}{t_6} & \frac{t_3}{t_1} & 0 \\ 0 & 1 & \frac{t_3 t_6}{t_1} & -\frac{1}{t_4} \\ 0 & 0 & 1 & \frac{1}{t_3 t_4} \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \frac{1}{t_4 t_6} & 0 & 0 & 0 \\ 0 & \frac{t_6}{t_1} & 0 & 0 \\ 0 & 0 & \frac{t_1}{t_3} & 0 \\ 0 & 0 & 0 & t_3 t_4 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 & 0 \\ -\frac{t_1 + t_6}{t_4 t_6} & 1 & 0 & 0 \\ \frac{1}{t_1 t_4} & -\frac{1}{t_1} & 1 & 0 \\ -\frac{1}{t_3 t_4} & \frac{1}{t_3} & -\frac{t_1}{t_3} & 1 \end{pmatrix}.$$

Thus we have

$$\vec{y} = [g^T \dot{w}]_L^+ = \begin{pmatrix} 1 & \frac{1}{t_6} & \frac{t_3}{t_1} & 0 \\ 0 & 1 & \frac{t_3 t_6}{t_1} & -\frac{1}{t_4} \\ 0 & 0 & 1 & \frac{1}{t_3 t_4} \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad and \quad \dot{v}\vec{y} = \begin{pmatrix} 0 & 0 & 1 & \frac{1}{t_3 t_4} \\ -1 & -\frac{1}{t_6} & -\frac{t_3}{t_1} & 0 \\ 0 & -1 & -\frac{t_3 t_6}{t_1} & \frac{1}{t_4} \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The top-aligned flag minors of $\dot{v}\vec{y}$ give the pre-twist $\vec{\tau}_{v,w}^{\mathrm{pre}}(gB_-) = B_- v\vec{y}$. To apply the chiral map to $B_- v\vec{y}$, we need to apply downward row operations to $\dot{v}\vec{y} \in (\dot{v}N)^{\cap w}$ to get an element $y_1 \in (N\dot{v} \cap \dot{v}N)^{\cap w}$. This element is given by

$$y_1 = \begin{pmatrix} 0 & 0 & 1 & \frac{1}{t_3 t_4} \\ -1 & -\frac{1}{t_6} & 0 & \frac{1}{t_1 t_4} \\ 0 & -1 & 0 & \frac{t_1 + t_6}{t_1 t_4} \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad thus \quad \vec{\tau}_{v,w}(gB_-) = \vec{\chi}_v(B_- v\vec{y}) = \vec{\chi}_v(B_- y_1) = y_1 B_-.$$

Notice that each right-aligned flag minor of y_1 is positive (and in fact, subtraction-free as a function of t_1, t_3, t_4, t_6). We may also compute the left twist: we have $\overleftarrow{\chi}_v(gB_-) = B_- g_\cap$,

$$\overleftarrow{y} = [\dot{w}g_\cap^T]_R^+ = \begin{pmatrix} 1 & \frac{t_1}{t_3} & \frac{1}{t_3} & \frac{1}{t_3 t_4} \\ 0 & 1 & \frac{1}{t_1} & \frac{1}{t_1 t_4} \\ 0 & 0 & 1 & \frac{t_1 + t_6}{t_4 t_6} \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad and \quad \overleftarrow{\tau}_{v,w}(gB_-) = \overleftarrow{y}vB_- = \begin{pmatrix} -\frac{t_1}{t_3} & -\frac{1}{t_3} & 1 & \frac{1}{t_3 t_4} \\ -1 & -\frac{1}{t_1} & 0 & \frac{1}{t_1 t_4} \\ 0 & -1 & 0 & \frac{t_1 + t_6}{t_4 t_6} \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot B_-.$$

In the (a, b, c, d)-coordinates in (2.4), the right and the left twist maps are given by

$$(a, b, c, d) \xmapsto{\vec{\tau}_{v,w}} \left(\frac{1}{a}, \frac{1}{b}, \frac{d}{b}, \frac{c}{cd - b} \right), \quad (a, b, c, d) \xmapsto{\overleftarrow{\tau}_{v,w}} \left(\frac{1}{a}, \frac{1}{b}, \frac{d}{cd - b}, \frac{c}{b} \right).$$
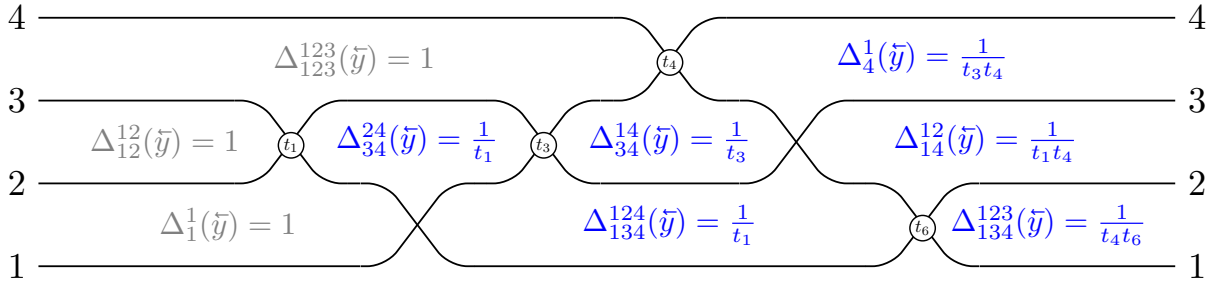
These two maps are mutually inverse and preserve total positivity; cf. (2.5). Finally, we check and compare the two Chamber Ansatz formulas (Theorem 2.15) in Figure 1.

(1.1), (1.4)	$\vec{R}^\circ_{v,w} := (B_- w B_- \cap B v B_-)/B_-, \quad \overleftarrow{R}^\circ_{v,w} := B_- \backslash (B_- w B_- \cap B_- v B)$
(2.2)	**Chiral map** $\vec{\chi}_v$: $\overleftarrow{R}^\circ_{v,w} \xleftarrow{\;\sim\;} (N\dot{v} \cap \dot{v}N)^{\cap w} \xrightarrow{\;\sim\;} \vec{R}^\circ_{v,w}$ $\qquad\qquad\qquad\qquad \underset{\vec{\chi}_v}{}$
	Marsh–Rietsch parametrization and (pre-)twist maps:
(2.8)	$g = \mathbf{g_{v,w}(t)} = g_1 \cdots g_m \in (N\dot{v})^{\cap w}, \quad g_r = \dot{s}_{i_r} \text{ if } r \notin J^\circ_{\mathbf{v}}, \quad g_r = x_{i_r}(t_r) \text{ if } r \in J^\circ_{\mathbf{v}}$
Thm. 2.12	$g_\cap \in (N\dot{v} \cap \dot{v}N)^{\cap w} \quad \text{satisfies} \quad g B_- = g_\cap B_-$
Dfn. 2.5	$\vec{y} = [g^T \dot{w}]^+_L, \quad \vec{\tau}^{\mathrm{pre}}_{v,w}(g B_-) = B_- v [g^T \dot{w}]^+_L, \quad \vec{\tau}_{v,w} = \vec{\chi}_v \circ \vec{\tau}^{\mathrm{pre}}_{v,w}$ $\overleftarrow{y} = [\dot{w} g^T_\cap]^+_R, \quad \overleftarrow{\tau}^{\mathrm{pre}}_{v,w}(B_- h) = [\dot{w} h^T]^+_R v B_-, \quad \overleftarrow{\tau}_{v,w} = \overleftarrow{\tau}^{\mathrm{pre}}_{v,w} \circ \vec{\chi}_v$
Thms. 2.3, 2.6	**The maps $\vec{\chi}_v$, $\vec{\tau}^{\mathrm{pre}}_{v,w}$, and $\vec{\tau}_{v,w}$ preserve total positivity**
	Chamber Ansatz formulas:
(2.9), (2.12)	$\vec{f}_r = \Delta^{s^{\mathbf{v}}_{i_m} \cdots s^{\mathbf{v}}_{i_r} \omega_{i_r}}_{s_{i_m} \cdots s_{i_r} \omega_{i_r}}, \quad \overleftarrow{f}_r = \Delta^{s_{i_1} \cdots s_{i_r} w_0 \omega_{i^*_r}}_{s^{\mathbf{v}}_{i_1} \cdots s^{\mathbf{v}}_{i_r} w_0 \omega_{i^*_r}}$
Thm. 2.8	Right twist Chamber Ansatz: express t_r in terms of $\vec{f}_j(\vec{y})$
Thm. 2.12	Left twist Chamber Ansatz: express t_r in terms of $\overleftarrow{f}_j(\overleftarrow{y})$
Thm. 5.1	Marsh–Rietsch Chamber Ansatz: express t_r in terms of $\overleftarrow{f}_j(z^T)$, where $z \in N_-$ satisfies $z w B_- = g B_-$
	Positivity-preserving involutions:
(3.2)	$x \mapsto x^T$: anti-automorphism, $x_i(t) \overset{T}{\leftrightarrow} y_i(t)$, $\dot{s}^T_i = \dot{s}^{-1}_i$; $a^T = a \ (a \in H)$
(3.3)	$x \mapsto x^\iota$: anti-automorphism, preserves $x_i(t)$, $y_i(t)$, $\dot{s}_i$; $a^\iota = a^{-1} \ (a \in H)$
(3.4)	$x \mapsto x^\theta = (x^\iota)^T$: automorphism, $x_i(t) \overset{\theta}{\leftrightarrow} y_i(t)$, $\dot{s}^\theta_i = \dot{s}^{-1}_i$; $a^\theta = a^{-1} \ (a \in H)$
	Generalized minors:
(3.7)	$\overline{\overline{w}} = \dot{w} = \dot{s}_{i_1} \cdots \dot{s}_{i_m}, \quad \overline{w} = \dot{s}^{-1}_{i_1} \cdots \dot{s}^{-1}_{i_m}$
(3.9), (3.11)	$\Delta^{v\omega_i}_{w\omega_i}(x) = \Delta^{\omega_i}_{\omega_i}\left(\overline{\overline{v^{-1}}} x \overline{w}\right) = \Delta^{w w_0 \omega_{i^*}}_{v w_0 \omega_{i^*}}(x^\iota)$
	Collision moves:
	$t_+ = 1/t, \quad a_+ = t^{\alpha^\vee_i}, \quad t_- = -1/t$
(3.17)	$\dot{s}^{-1}_i x_i(t) = x_i(t_-) a^{-1}_+ y_i(t_+), \quad x_i(t) \dot{s}^{-1}_i = y_i(t_+) a_+ x_i(t_-)$
(3.18)	$\dot{s}_i y_i(t) = y_i(t_-) a_+ x_i(t_+), \quad y_i(t) \dot{s}_i = x_i(t_+) a^{-1}_+ y_i(t_-)$
(7.1), (8.6)	$g^T \dot{w} = y_+ y_0 y_-, \quad \vec{y} = y_+, \quad z = (\dot{w} y_- \dot{w}^{-1})^T$

TABLE 1. Useful notation and formulas.
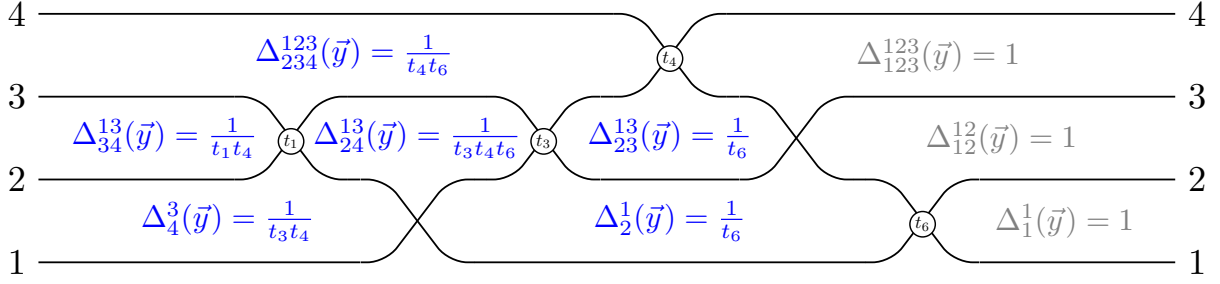
(a) Right twist Chamber Ansatz: $\vec{f}_j(\vec{y}) = \vec{\Delta}_{C_r^{\blacktriangleright}}(\vec{y}) = \Delta^{I_{\rightarrow}^{\vee}(C_r^{\blacktriangleright})}_{I_{\rightarrow}^{\mathbf{w}}(C_r^{\blacktriangleright})}(\vec{y})$.

(b) Left twist Chamber Ansatz: $\overleftarrow{f}_j(\breve{y}) = \overleftarrow{\Delta}_{C_r^{\blacktriangleleft}}(\breve{y}) = \Delta^{I_{\mathbf{w}}^{\leftarrow}(C_r^{\blacktriangleleft})}_{I_{\mathbf{v}}^{\leftarrow}(C_r^{\blacktriangleleft})}(\breve{y})$.

FIGURE 1. Comparing the Chamber Ansatz formulas (Theorems 2.8 and 2.12). See also Theorem 2.15.

3. BACKGROUND

3.1. **Pinnings.** Our exposition mostly follows that of [GKL21]. We denote by $X(H) :=$ $\mathrm{Hom}(H, \mathbb{C}^*)$ the *weight lattice* and for $\gamma \in X(H)$ and $a \in H$, we let a^{γ} denote the value of γ on a. Recall that we have the root system $\Phi \subset X(H)$ of G and a set of simple roots $\{\alpha_i\}_{i \in I}$. Let $Y(H) := \mathrm{Hom}(\mathbb{C}^*, H)$ be the *coweight lattice*. We have simple coroots $\{\alpha_i\}_{i \in I} \subset Y(H)$ given by $t^{\alpha_i^{\vee}} := \phi_i \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \in Y(H)$. Consider the natural pairing $\langle \cdot, \cdot \rangle : X(H) \times Y(H) \to \mathbb{Z}$ such that for all $\gamma \in X(H)$, $\beta \in Y(H)$, and $t \in \mathbb{C}^*$, we have $(t^{\beta})^{\gamma} = t^{\langle \gamma, \beta \rangle}$. Let $\{\omega_i\}_{i \in I} \subset X(H)$ be the *fundamental weights*, defined so that $\langle \omega_i, \alpha_j^{\vee} \rangle = \delta_{i,j}$ for $i, j \in I$.

For $\gamma \in X(H)$, $t \in \mathbb{C}^*$, $a \in H$, and $w \in W$, we have [FZ02, Equations (1.2) and (2.5)]

$$(3.1) \qquad (\dot{w} a \dot{w}^{-1})^{\gamma} = a^{w\gamma}, \quad a x_i(t) a^{-1} = x_i(a^{\alpha_i} t), \quad a y_i(t) a^{-1} = y_i(a^{-\alpha_i} t).$$

The totally positive part $H_{>0}$ of H is the subgroup generated by $t^{\alpha_i^{\vee}}$ for $i \in I$ and $t \in \mathbb{R}_{>0}$.

3.2. **Involutions.** We will make use of three involutions on G; cf. [FZ99, Section 2.1]. We have two involutive anti-automorphisms $x \mapsto x^T$ and $x \mapsto x^{\iota}$ of G and one involutive automorphism $x \mapsto x^{\theta} = (x^T)^{\iota} = (x^{\iota})^T$ defined by

$$(3.2) \qquad a^T = a \quad (a \in H), \qquad x_i(t)^T = y_i(t), \qquad y_i(t)^T = x_i(t);$$

$$(3.3) \qquad a^{\iota} = a^{-1} \quad (a \in H), \qquad x_i(t)^{\iota} = x_i(t), \qquad y_i(t)^{\iota} = y_i(t);$$

$$(3.4) \qquad a^{\theta} = a^{-1} \quad (a \in H), \qquad x_i(t)^{\theta} = y_i(t), \qquad y_i(t)^{\theta} = x_i(t).$$

Remark 3.1. Each of these three involutions clearly preserves the subset $G_{\geq 0} \subset G$. Thus we have involutive homeomorphisms

$$(3.5) \qquad (G/B_-)_{\geq 0} \xrightarrow{\sim} (B_-\backslash G)_{\geq 0}, \qquad\qquad gB_- \mapsto B_- g^\iota;$$

$$(3.6) \qquad (G/B)_{\geq 0} \xrightarrow{\sim} (G/B_-)_{\geq 0}, \qquad\qquad gB \mapsto g^\theta B_-.$$

We always apply the map (3.6) to the objects in [MR04, Ing19] in order to switch from G/B to G/B_-.

The above involutions commute pairwise and they also commute with $x \mapsto x^{-1}$. We have

$$\dot{s}_i^\iota = \dot{s}_i = \phi_i \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \text{and} \quad \dot{s}_i^T = \dot{s}_i^\theta = \dot{s}_i^{-1} = \phi_i \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

For $w \in W$, choose a reduced expression $\mathbf{w} = (i_1, \ldots, i_m)$ and let

$$(3.7) \qquad \overline{\overline{w}} := \dot{w} = \dot{s}_{i_1} \cdots \dot{s}_{i_m}, \quad \overline{w} := ((w^{-1})^\cdot)^{-1} = \dot{s}_{i_1}^{-1} \cdots \dot{s}_{i_m}^{-1}.$$

Using this notation, we find [FZ99, Proposition 2.1]

$$\dot{w}^\iota = \overline{\overline{w^{-1}}}, \quad \dot{w}^T = \dot{w}^{-1} = \overline{w^{-1}}, \quad \text{and} \quad \dot{w}^\theta = \overline{w}.$$

3.3. Generalized minors.

The following construction appears in [FZ99, Section 1.4]. Recall that we set $G_0^\pm := BB_-$ and $G_0^\mp := B_-B$. For $i \in I$, let $\Delta_{\omega_i}^{\omega_i}$ be the regular function whose restriction to $x \in G_0^\mp$ is given by

$$\Delta_{\omega_i}^{\omega_i}(x) := ([x]_0^\mp)^{\omega_i}.$$

In particular, for $x \in G$, $y_- \in N_-$, and $y_+ \in N$, we have

$$(3.8) \qquad \Delta_{\omega_i}^{\omega_i}(y_- x y_+) = \Delta_{\omega_i}^{\omega_i}(x).$$

For $i \in I$, $v, w \in W$, and $x \in G$, we set

$$(3.9) \qquad \Delta_{w\omega_i}^{v\omega_i}(x) := \Delta_{\omega_i}^{\omega_i}\left(\overline{\overline{v^{-1}}} x \overline{w}\right).$$

For $a \in H$, we have [FZ99, Equation (2.14)]

$$(3.10) \qquad \Delta_{w\omega_i}^{v\omega_i}(a \cdot x) = \Delta_{v\omega_i}^{v\omega_i}(a) \cdot \Delta_{w\omega_i}^{v\omega_i}(x).$$

We also have the following identity [FZ99, Equation (2.25)]:

$$(3.11) \qquad \Delta_{w\omega_i}^{v\omega_i}(x) = \Delta_{v\omega_i}^{w\omega_i}(x^T) = \Delta_{vw_0\omega_{i*}}^{ww_0\omega_{i*}}(x^\iota).$$

3.4. Factorizations.

For $u \in W$, we define two subgroups

$$(3.12) \qquad N(u) := N \cap \dot{u}^{-1} N_- \dot{u} \quad \text{and} \quad N'(u) := N \cap \dot{u}^{-1} N \dot{u}.$$

These subgroups do not depend on the choice of the representative $\dot{u}$ of u inside $N_G(H)$. The following result is well known.

Lemma 3.2. *The multiplication map provides isomorphisms*

$$N(u) \times N'(u) \xrightarrow{\sim} N, \quad N'(u) \times N(u) \xrightarrow{\sim} N, \quad \text{and} \quad (\dot{u}^{-1} N \dot{u} \cap N) \times (\dot{u}^{-1} N \dot{u} \cap N_-) \xrightarrow{\sim} \dot{u}^{-1} N \dot{u}.$$

Lemma 3.3 ([Lec16]). *Let* $v \leq w \in W$ *and let* $(\mathbf{v}, \mathbf{w}) \in \mathrm{Red}(v, w)$. *The functions* $\vec{f}_r$ *defined in* (2.9) *belong to the doubly-invariant ring* $^{N(v)}\mathbb{C}[N]^{N'(w)}$. *In other words, for all* $(y, y') \in N(v) \times N'(w)$ *and* $x \in N$, *we have*

$$\vec{f}_r(yxy') = \vec{f}_r(x).$$

Our next result gives an explicit procedure for converting an MR-parametrized element $g \in N\dot{v}$ such that $gB_- \in \vec{R}^{\circ}_{v,w}$ into an element $g_{\cap} \in N\dot{v} \cap \dot{v}N$ satisfying $gB_- = g_{\cap}B_-$; cf. (2.2).

Lemma 3.4. *For any* $g \in N\dot{v}$, *there exists a unique element* $x \in N_- \cap \dot{v}^{-1}N\dot{v}$ *such that*

$$g_{\cap} := gx \in N\dot{v} \cap \dot{v}N.$$

Proof. Using Lemma 3.2, we factorize $\dot{v}^{-1}g \in \dot{v}^{-1}N\dot{v}$ as a product y_1y_2 with $y_1 \in \dot{v}^{-1}N\dot{v} \cap N$ and $y_2 \in \dot{v}^{-1}N\dot{v} \cap N_-$. Setting $g_{\cap} := \dot{v}y_1$ and $x := y_2^{-1}$ shows the existence part. Since the procedure can be reversed, the uniqueness part follows. □

The next lemma is well known; see e.g. [GKL21, Equation (4.33)].

Lemma 3.5. *Let* $v, w \in W$. *We have the following isomorphisms:*

(3.13) $N_-\dot{w} \cap \dot{w}N \xrightarrow{\sim} (B_-wB_-)/B_-,$ $g \mapsto gB_-;$

(3.14) $\dot{w}N_- \cap N\dot{w} \xrightarrow{\sim} B_-\backslash(B_-wB_-),$ $g \mapsto B_-g;$

Lemma 3.6. *For* $w \in W$, $b \in B$, *and* $b_- \in B_-$, *we have* $\dot{w}b\dot{w}^{-1} \in G_0^{\mp}$, $\dot{w}^{-1}b_-\dot{w} \in G_0^{\pm}$,

(3.15) $[\dot{w}b\dot{w}^{-1}]_R^+ \in \dot{w}N\dot{w}^{-1} \cap N, \quad and \quad [\dot{w}^{-1}b_-\dot{w}]_L^+ \in \dot{w}^{-1}N_-\dot{w} \cap N.$

In particular, for $g \in B_-wB_-$, *we have*

(3.16) $\dot{w}^{-1}g \in G_0^{\pm}, \quad and \quad [\dot{w}^{-1}g]_L^+ \in \dot{w}^{-1}N_-\dot{w} \cap N;$

Proof. Let $b = ay$ with $a \in H$ and $y \in N$. Using Lemma 3.2, we factorize $y = y_1y_2$ with $y_1 \in N \cap \dot{w}^{-1}N_-\dot{w}$ and $y_2 \in N \cap \dot{w}^{-1}N\dot{w}$. Then

$$\dot{w}b\dot{w}^{-1} = \dot{w}ay_1\dot{w}^{-1} \cdot \dot{w}y_2\dot{w}^{-1} \in B_- \cdot (\dot{w}N\dot{w}^{-1} \cap N).$$

This shows the first part of (3.15), and the second part follows by a similar argument.

To show (3.16), let $b_- \in B_-$ be such that $g \in b_-wB_-$. By (3.15), we have $\dot{w}^{-1}b_-\dot{w} \in (\dot{w}^{-1}N_-\dot{w} \cap N) \cdot B_-$, and thus

$$\dot{w}^{-1}g \in \dot{w}^{-1}b_-\dot{w}B_- \subset (\dot{w}^{-1}N_-\dot{w} \cap N) \cdot B_-. \qquad \Box$$

3.5. **Collision moves.** By [FZ02, Equation (2.13)], for each $t \in \mathbb{C}^*$ and $i \in I$, there exist $t_+ = 1/t$, $a_+ = t^{\alpha_i^{\vee}} \in H$, and $t_- = -1/t$ satisfying

(3.17) $\dot{s}_i^{-1}x_i(t) = x_i(t_-)a_+^{-1}y_i(t_+), \quad x_i(t)\dot{s}_i^{-1} = y_i(t_+)a_+x_i(t_-),$

(3.18) $\dot{s}_iy_i(t) = y_i(t_-)a_+x_i(t_+), \quad y_i(t)\dot{s}_i = x_i(t_+)a_+^{-1}y_i(t_-).$

Note that if $t > 0$ then $t_+ > 0$, $a_+ \in H_{>0}$, and $t_- < 0$.

We also record the following *conjugation moves*.

Lemma 3.7. *Let* $w \in W$, $i \in I$, *and* $t \in \mathbb{C}^*$.

    (i) *If* $ws_i < w$ *then* $\dot{w}x_i(t)\dot{w}^{-1} \in N_-$ *and* $\dot{w}y_i(t)\dot{w}^{-1} \in N$.

    (ii) *If* $ws_i > w$ *then* $\dot{w}x_i(t)\dot{w}^{-1} \in N$ *and* $\dot{w}y_i(t)\dot{w}^{-1} \in N_-$.

(iii) *For all $h \in N'(s_i)$, we have $y_i(t)hy_i(-t) \in N$.*

(iv) *For all $h \in N$, we have $\dot{s}_i^{-1}h\dot{s}_i = h'y_i(t')$ for some $h' \in N$ and $t' \in \mathbb{C}$.*

(v) *We have $\dot{w}_0 x_{i*}(-t)\dot{w}_0^{-1} = y_i(t)$.*

Proof. Parts (i) and (ii) are well known and may be deduced e.g. from [GKL21, Section 4.2]. Part (iii) is also well known and follows from [GKL21, Lemma 4.3]. For part (iv), we use Lemma 3.2 to factorize $h = h_1 h_2$ with $h_1 \in N \cap \dot{s}_i N \dot{s}_i^{-1}$ and $h_2 \in N \cap \dot{s}_i N_- \dot{s}_i^{-1}$. Then $h' := \dot{s}_i^{-1} h_1 \dot{s}_i \in N$ and $\dot{s}_i^{-1} h_2 \dot{s}_i = y_i(t')$, as expected.

For part (v), it is clear that we have $\dot{w}_0 x_{i*}(-t)\dot{w}_0^{-1} = y_i(t')$ for some $t' \in \mathbb{C}$. Applying (3.9), we get

$$\Delta_{\omega_i}^{s_i\omega_i}(\dot{w}_0 x_{i*}(-t)\dot{w}_0^{-1}) = \Delta_{\omega_i}^{\omega_i}(\dot{s}_i \dot{w}_0 x_{i*}(-t)\dot{w}_0^{-1}) = \Delta_{\omega_i}^{\omega_i}(\dot{w}_0 \dot{s}_{i*} x_{i*}(-t)\dot{w}_0^{-1}).$$

Taking the inverse of the second equation in (3.17), we find $\dot{s}_{i*} x_{i*}(-t) = x_{i*}(t_+)a_+^{-1} y_{i*}(t_-)$, where $a_+ = t^{\alpha_{i*}^\vee}$, $t_+ = 1/t$, and $t_- = -1/t$. Applying part (ii), we find that

$$\dot{w}_0 x_{i*}(t_+)a_+^{-1} y_{i*}(t_-)\dot{w}_0^{-1} \in N_- \dot{w}_0 a_+^{-1} \dot{w}_0^{-1} N.$$

By (3.1) and (3.8), we get

$$\Delta_{\omega_i}^{\omega_i}(\dot{w}_0 \dot{s}_{i*} x_{i*}(-t)\dot{w}_0^{-1}) = \Delta_{\omega_i}^{\omega_i}(\dot{w}_0 a_+^{-1} \dot{w}_0^{-1}) = (\dot{w}_0 a_+^{-1} \dot{w}_0^{-1})^{\omega_i} = t^{\langle \omega_{i*}, \alpha_{i*}^\vee \rangle} = t.$$

On the other hand, by (3.18) and (3.8),

$$\Delta_{\omega_i}^{s_i\omega_i}(y_i(t)) = \Delta_{\omega_i}^{\omega_i}(\dot{s}_i y_i(t)) = \Delta_{\omega_i}^{\omega_i}(y_i(t_-)t^{\alpha_i^\vee} x_i(t_+)) = \Delta_{\omega_i}^{\omega_i}(t^{\alpha_i^\vee}) = t^{\langle \omega_i, \alpha_i^\vee \rangle} = t.$$

This finishes the proof of part (v). $\square$

4. Basic properties of the twist

The goal of this section is to prove that $\vec{\tau}_{v,w}$ gives a well-defined automorphism of $\vec{R}_{v,w}^\circ$; see Corollary 4.3. Since $\vec{\tau}_{v,w}$ is defined as the composition of the chiral map $\vec{\chi}_v$ and the pre-twist $\vec{\tau}_{v,w}^{\mathrm{pre}}$, we need to study these maps first. Recall that the pre-twist is induced by the map $g \mapsto \dot{v}[g^T \dot{w}]_L^+$ introduced in Proposition 2.4.

Proof of Proposition 2.4. Let $g \in (N\dot{v})^{\cap w}$. Since $g \in B_- w B_-$, by (3.16), we have $\dot{w}^{-1}g \in G_0^\pm$, and thus $g^T \dot{w} \in G_0^\pm$. In particular, $[g^T \dot{w}]_L^+$ is well defined. Set $h := \dot{v}[g^T \dot{w}]_L^+ \in \dot{v}N$. By definition, $[g^T \dot{w}]_L^+ \in g^T \dot{w} \cdot B_-$. Since $g \in N\dot{v}$, we find

$$h = \dot{v}[g^T \dot{w}]_L^+ \in \dot{v}g^T \dot{w} \cdot B_- \subset \dot{v}(\dot{v}^{-1}N_-)\dot{w} \cdot B_- \subset B_- w B_-.$$

We have shown that $h \in (\dot{v}N)^{\cap w}$. A similar argument shows that $[\dot{w}h^T]_R^+$ is well defined and that $[\dot{w}h^T]_R^+ \dot{v} \in (N\dot{v})^{\cap w}$. It remains to show that the two maps are inverses of each other, i.e., that $g = [\dot{w}h^T]_R^+ \dot{v}$.

Write

$$\dot{w}^{-1}g = g_+ g_0 g_-, \quad \text{where} \quad (g_+, g_0, g_-) \in N \times H \times N_-.$$

By (3.13), we have $g_+ \in \dot{w}^{-1}N_- \dot{w} \cap N$. By definition, we have $h = \dot{v}g^T$. We find

$$\dot{w}h^T = \dot{w}g_- \dot{v}^{-1} = \dot{w}(g_0^{-1} g_+^{-1} \dot{w}^{-1} g)\dot{v}^{-1} = (\dot{w}g_0^{-1} \dot{w}^{-1}) \cdot (\dot{w}g_+^{-1} \dot{w}^{-1}) \cdot (g\dot{v}^{-1}).$$

Since $g_0 \in H$, we have $\dot{w}g_0^{-1}\dot{w}^{-1} \in H$. Since $g_+ \in \dot{w}^{-1}N_- \dot{w} \cap N$, we have $\dot{w}g_+^{-1}\dot{w}^{-1} \in N_-$. Thus, $\dot{w}h^T \in B_- \cdot (g\dot{v}^{-1})$. Since $g \in N\dot{v}$, we have $[\dot{w}h^T]_R^+ = g\dot{v}^{-1}$, so indeed $[\dot{w}h^T]_R^+ \dot{v} = g$. $\square$

Our next result will be used to show that the right pre-twist $\vec{\tau}_{v,w}^{\mathrm{pre}}$ is well defined.

Lemma 4.1. *Let $g_1, g_2 \in (N\dot{v})^{\cap w}$ be such that $g_1 B_- = g_2 B_-$. Let $h_1 := \dot{v}[g_1^T \dot{w}]_L^+$ and $h_2 := \dot{v}[g_2^T \dot{w}]_L^+$. Then $B_- h_1 = B_- h_2$.*

Proof. Since $g_1, g_2 \in N\dot{v}$ and $g_1 B_- = g_2 B_-$, by Lemma 3.4, we have $g_2 = g_1 x$ for some $x \in \dot{v}^{-1} N\dot{v} \cap N_-$. Since $x^T \in N$, we have

$$h_2 = \dot{v}[g_2^T \dot{w}]_L^+ = \dot{v}[x^T g_1^T \dot{w}]_L^+ = \dot{v}x^T[g_1^T \dot{w}]_L^+ = \dot{v}x^T \dot{v}^{-1} \cdot \dot{v}[g_1^T \dot{w}]_L^+ = \dot{v}x^T \dot{v}^{-1} \cdot h_1.$$

Since $x^T \in \dot{v}^{-1} N_- \dot{v}$, we find $\dot{v}x^T \dot{v}^{-1} \in N_-$, and thus $h_2 \in B_- h_1$. $\square$

Corollary 4.2. *The right and left pre-twists $\vec{\tau}_{v,w}^{\mathrm{pre}}$ and $\overleftarrow{\tau}_{v,w}^{\mathrm{pre}}$ are well-defined mutually inverse isomorphisms between $\vec{R}_{v,w}^\circ$ and $\overleftarrow{R}_{v,w}^\circ$.*

Proof. Each point of $\vec{R}_{v,w}^\circ$ is of the form gB_- for some $g \in (N\dot{v})^{\cap w}$ in view of (2.2). However, the projection map $(N\dot{v})^{\cap w} \to \vec{R}_{v,w}^\circ$, $g \mapsto gB_-$ is surjective but in general not injective. Lemma 4.1 shows that $\vec{\tau}_{v,w}^{\mathrm{pre}}(gB_-)$ does not depend on the choice of the representative $g \in (N\dot{v})^{\cap w}$, and thus $\vec{\tau}_{v,w}^{\mathrm{pre}}$ is well defined on $\vec{R}_{v,w}^\circ$. By Proposition 2.4, its image is contained inside $\overleftarrow{R}_{v,w}^\circ$. A similar argument shows that the left pre-twist $\overleftarrow{\tau}_{v,w}^{\mathrm{pre}}$ is well defined on $\overleftarrow{R}_{v,w}^\circ$ and has image contained inside $\vec{R}_{v,w}^\circ$. It follows from Proposition 2.4 that the two maps are mutually inverse isomorphisms. $\square$

It follows from (2.2) that the chiral map $\vec{\chi}_v$ restricts to an isomorphism $\overleftarrow{R}_{v,w}^\circ \xrightarrow{\sim} \vec{R}_{v,w}^\circ$, and thus we get the following result.

Corollary 4.3. *The right and left twist maps $\vec{\tau}_{v,w}$ and $\overleftarrow{\tau}_{v,w}$ are well-defined mutually inverse automorphisms of $\vec{R}_{v,w}^\circ$.*

5. Marsh–Rietsch Chamber Ansatz

Our goal is to recall the results of [MR04] and connect them to our Chamber Ansatz formulas. The following result appears in [MR04, Theorem 7.1(1)]. We shall compare it to Theorem 2.12.

Theorem 5.1 (Marsh–Rietsch Chamber Ansatz [MR04]). *Let $v \leq w$ in W and $(\mathbf{v}, \mathbf{w}) \in \mathrm{Red}(v, w)$. Let $g := \mathbf{g}_{\mathbf{v}, \mathbf{w}}(\mathbf{t}) \in (N\dot{v})^{\cap w}$ be MR-parametrized, and let $z \in N_-$ be any element satisfying $gB_- = zwB_-$. Then for all $r \in J_{\mathbf{v}}^\circ$, we have*

$$(5.1) \qquad t_r = \frac{\prod_{j \neq i_r^*} \Delta_{v_{(r)} w_0 \omega_j}^{w_{(r)} w_0 \omega_j}(z^T)^{-a_{j,i_r^*}}}{\Delta_{v_{(r)} w_0 \omega_{i_r^*}}^{w_{(r)} w_0 \omega_{i_r^*}}(z^T) \Delta_{v_{(r-1)} w_0 \omega_{i_r^*}}^{w_{(r-1)} w_0 \omega_{i_r^*}}(z^T)}.$$

Conversely, for each $j \in [m]$, we have

$$(5.2) \qquad \overleftarrow{f}_j(z^T) = \prod_{r \in J_{\mathbf{v}}^\circ : r \leq j} t_r^{-\langle \omega_{i_j}, s_{i_j} s_{i_{j-1}} \cdots s_{i_{r+1}} \alpha_{i_r}^\vee \rangle}.$$

Proof of Theorem 2.12. Let $g_\cap \in (N\dot{v} \cap \dot{v}N)^{\cap w}$ and $\overleftarrow{y} := [\dot{w}g_\cap^T]_R^+$ be as in Theorem 2.12. By [MR04, Theorem 7.1], the right-hand side of (5.1) does not depend on the choice of the element $z \in N_-$ satisfying $gB_- = zwB_-$. Our goal is to show that the element $z := \overleftarrow{y}^T$ satisfies these assumptions. Clearly, $\overleftarrow{y} \in N$, so $z \in N_-$. We concentrate on showing that $gB_- = \overleftarrow{y}^T wB_-$.

Since $g_\cap \in B_- w B_-$, by (3.16), we have $\dot{w}^{-1} g_\cap \in G_0^\pm$, and thus $g_\cap^T \dot{w} \in G_0^\pm$. Moreover, (3.16) implies that

$$(5.3) \qquad\qquad g_\cap^T \dot{w} = b y_- \quad \text{for some } b \in B \text{ and } y_- \in \dot{w}^{-1} N \dot{w} \cap N_-.$$

By definition, $g_\cap B_- = g B_-$. Thus

$$(5.4) \qquad\qquad\qquad B g^T = B g_\cap^T = B y_- \dot{w}^{-1}.$$

On the other hand, conjugating (5.3) by $\dot{w}$ and using $\dot{w} y_- \dot{w}^{-1} \in N$, we find

$$\dot{w} g_\cap^T = \dot{w} b \dot{w}^{-1} \cdot \dot{w} y_- \dot{w}^{-1}, \quad \breve{y} = [\dot{w} g_\cap^T]_R^+ = [\dot{w} b \dot{w}^{-1}]_R^+ \cdot \dot{w} y_- \dot{w}^{-1}.$$

Let $b' := [\dot{w} b \dot{w}^{-1}]_R^+$. By (3.15), we have $b' \in \dot{w} N \dot{w}^{-1} \cap N$. In particular, $B w^{-1} b' = B w^{-1}$. Therefore

$$(5.5) \qquad\qquad B y_- \dot{w}^{-1} = B w^{-1} \cdot \dot{w} y_- \dot{w}^{-1} = B w^{-1} b' \cdot \dot{w} y_- \dot{w}^{-1} = B w^{-1} \breve{y}.$$

Combining (5.4) with (5.5), we get $B g^T = B w^{-1} \breve{y}$, and thus $g B_- = \breve{y}^T w B_-$. $\qquad\square$

6. CHIRAL AND REVERSAL MAPS

The goal of Section 6.1 is to prove Theorem 2.3. In Section 6.2, we study *reversal maps* which swap the roles of (v, w) and $(w w_0, v w_0)$. In Section 6.3, we use them to introduce the *opposite chiral map* which will be later used in Section 7 to show that the twist map preserves total positivity.

6.1. **The chiral map preserves total positivity.**

Lemma 6.1. *Let $v \le w$ in W and $g B_- \in \vec{R}_{v,w}^{>0}$. Let $i \in I$ be such that $s_i v < v$. Then*

$$y_i(t) g B_- \in \vec{R}_{s_i v, w}^{>0} \quad \text{for all } t > 0.$$

Proof. For $t > 0$, we have $y_i(t) \in G_{\ge 0}$, and thus $y_i(t) \cdot (G/B_-)_{\ge 0} \subset (G/B_-)_{\ge 0}$; cf. [Lus94]. Since $y_i(t) \in B_-$ and $g \in B_- w B_-$, we have $y_i(t) g \in B_- w B_-$. Since $y_i(t) \in B s_i B$ and $g \in B v B_-$, by the properties of the Bruhat decomposition, we have $y_i(t) g \in B s_i v B_-$ when $s_i v < v$. By definition, $\vec{R}_{s_i v, w}^{>0} = (G/B_-)_{\ge 0} \cap B s_i v B_- \cap B_- w B_-$. $\qquad\square$

We will need the following technical result.

Lemma 6.2. *Let $v, u_1, u_2 \in W$ be such that $\ell(u_1 v) = \ell(u_1) + \ell(v)$ and $\ell(v u_2) = \ell(v) + \ell(u_2)$. Let $w := v u_2$ and let $\mathbf{w}$ be a reduced expression for w that starts with a reduced expression for v. Let $\mathbf{v}$ be the positive distinguished subexpression for v inside $\mathbf{w}$.[1] Choose an MR-parametrized element $g = \mathbf{g}_{\mathbf{v}, \mathbf{w}}(\mathbf{t}) \in (N \dot{v})^{\cap w}$ with $\mathbf{t} \in (\mathbb{R}_{>0})^{J_{\mathbf{v}}^\circ}$ and set*

$$\breve{g} := \dot{u}_1 \dot{v} [\dot{v}^{-1} g]_L^+.$$

Then $B_- \breve{g} \in (B_- \backslash G)_{\ge 0}$.

Proof. We prove the result by induction on $\ell(v)$.

Consider the base case $v = \mathrm{id}$. Then $\dot{v}^{-1} g \in N$ so $\breve{g} = \dot{u}_1 g$. The element g is a product of elements of the form $x_i(t) \in G_{\ge 0}$ whose right action preserves $(B_- \backslash G)_{\ge 0}$, and thus indeed $B_- \breve{g} = B_- \dot{u}_1 g \in (B_- \backslash G)_{\ge 0}$ since $B_- \dot{u}_1 \in (B_- \backslash G)_{\ge 0}$.

For the induction step, assume that $\ell(v) > 0$ and that the result has been shown for all v' satisfying $\ell(v') < \ell(v)$. Write $g = g_1 g_2 \cdots g_m$ as in (2.8). Suppose that the reduced

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[1]Note that $\mathbf{v}$ is the *rightmost* subexpression for v inside $\mathbf{w}$, while the start of $\mathbf{w}$ gives the *leftmost* subexpression for v, and the two subexpressions may overlap.

expressions for v and w start with s_i. We consider two cases: either $g_1 = \dot{s}_i$ or $g_1 = x_i(t)$. In each case, we will set $g' := g_2 \cdots g_m$, $v' := s_i v < v$, $u'_1 := u_1 s_i$, and $u'_2 := u_2$.

Assume first that $g_1 = \dot{s}_i$ for $i \in I$. Then $(\dot{v}')^{-1} g' = \dot{v} g$ and $\dot{u}'_1 \dot{v}' = \dot{u}_1 \dot{v}$, so

$$\bar{g}' := \dot{u}'_1 \dot{v}' [(\dot{v}')^{-1} g']_L^+ = \dot{u}_1 \dot{v} [\dot{v}^{-1} g]_L^+ = \bar{g}.$$

By the induction hypothesis, we have $B_- \bar{g} \in (B_- \backslash G)_{\geq 0}$.

Assume now that $g_1 = x_i(t)$, with $t > 0$. Then $\dot{v}^{-1}$ ends with $\dot{s}_i^{-1}$. Applying a collision move (3.17), we replace $\dot{s}_i^{-1} x_i(t)$ with $x_i(t_-) a_+ y_i(t_+)$ and get

$$(6.1) \qquad \dot{u}_1 \dot{v} [\dot{v}^{-1} g]_L^+ = \dot{u}_1 \dot{s}_i \dot{v}' [(\dot{v}')^{-1} \cdot x_i(t_-) a_+ y_i(t_+) \cdot g']_L^+.$$

First, we have $g' B_- \in \vec{R}^{>0}_{v, s_i w}$, and thus by Lemma 6.1, $y_i(t_+) g' B_- \in \vec{R}^{>0}_{s_i v, s_i w}$. Therefore $a_+ y_i(t_+) g' B_- \in \vec{R}^{>0}_{s_i v, s_i w}$. Recall that we set $v' := s_i v$ and $u'_2 := u_2$, and thus $w' := v' u'_2 = s_i w$. Choose a reduced expression $\mathbf{w}'$ for w' starting with a reduced expression for v' and let $g'' = \mathbf{g}_{\mathbf{v}', \mathbf{w}'}(\mathbf{t}')$ be the corresponding MR-parametrized element satisfying $a_+ y_i(t_+) g' B_- = g'' B_-$. Since $s_i v' > v'$, by Lemma 3.7(ii), we find $(\dot{v}')^{-1} x_i(t_-) \dot{v}' \in N$. Combining the pieces together, the right-hand side of (6.1) transforms into

$$\dot{u}_1 \dot{s}_i \dot{v}' [(\dot{v}')^{-1} x_i(t_-) \dot{v}' \cdot (\dot{v}')^{-1} \cdot g'']_L^+ = \dot{u}_1 \dot{s}_i \dot{v}' \cdot (\dot{v}')^{-1} x_i(t_-) \dot{v}' [(\dot{v}')^{-1} \cdot g'']_L^+ = \dot{u}_1 \dot{s}_i x_i(t_-) \dot{v}' [(\dot{v}')^{-1} \cdot g'']_L^+.$$

Since $\ell(u_1 v) = \ell(u_1) + \ell(v)$ and $s_i v < v$, we get $u_1 s_i > u_1$, and thus $\dot{u}_1 \dot{s}_i x_i(t_-) \in N_- \dot{u}_1 \dot{s}_i$ by Lemma 3.7(i). Thus

$$(6.2) \qquad B_- \dot{u}_1 \dot{s}_i x_i(t_-) \dot{v}' [(\dot{v}')^{-1} \cdot g'']_L^+ = B_- \dot{u}_1 \dot{s}_i \dot{v}' [(\dot{v}')^{-1} \cdot g'']_L^+ = B_- \dot{u}'_1 \dot{v}' [(\dot{v}')^{-1} \cdot g'']_L^+.$$

By the induction hypothesis, the right-hand side of (6.2) belongs to $(B_- \backslash G)_{\geq 0}$. $\square$

Proof of Theorem 2.3. We first prove the special case where $w = w_0$. Choose a reduced expression $\mathbf{w}_0 = (i_1, \ldots, i_m)$ for w_0 that starts with a reduced expression for v. Let $\mathbf{v}$ be the positive distinguished subexpression for v inside $\mathbf{w}_0$. Let $g := \mathbf{g}_{\mathbf{v}, \mathbf{w}_0}(\mathbf{t})$ be MR-parametrized. As we showed in the proof of Lemma 3.4, the element $g_\cap \in (N\dot{v} \cap \dot{v}N)^{\cap w_0}$ satisfying $g_\cap B_- = g B_-$ is given by

$$g_\cap = \dot{v} [\dot{v}^{-1} g]_L^+.$$

Applying Lemma 6.2 for $u_1 = \mathrm{id}$ and $u_2 = v^{-1} w_0$, we see that $B_- g_\cap \in (B_- \backslash G)_{\geq 0}$. We have shown that for $g B_- \in \vec{R}^{>0}_{v, w_0}$, the element $B_- g_\cap = \overleftarrow{\chi}_v(g B_-)$ belongs to $(B_- \backslash G)_{\geq 0}$. Clearly, $B_- g_\cap \in \overleftarrow{R}^{>0}_{v, w_0}$. The converse direction is handled similarly. We are done with the case $w = w_0$.

Let $v \leq w$ in W. Let $g_\cap \in (N\dot{v} \cap \dot{v}N)^{\cap w}$ be such that $g_\cap B_- \in \vec{R}^{>0}_{v, w}$. Then $g_\cap B_-$ belongs to the closure of $\vec{R}^{>0}_{v, w_0}$ inside $(G/B_-)_{\geq 0}$. The map $\overleftarrow{\chi}_v$ is continuous on the opposite Schubert cell $(BvB_-)/B_-$ which contains both $\vec{R}^{>0}_{v, w}$ and $\vec{R}^{>0}_{v, w_0}$. Therefore, $B_- g_\cap = \overleftarrow{\chi}_v(g_\cap B_-)$ belongs to the closure of $\overleftarrow{\chi}_v(\vec{R}^{>0}_{v, w_0}) = \overleftarrow{R}^{>0}_{v, w_0}$. In particular, $B_- g_\cap \in (B_- \backslash G)_{\geq 0}$. $\square$

6.2. Reversal maps.

Throughout this section, we fix a pair $v \leq w$ in W and set $v' := ww_0$, $w' := vw_0$. Our goal is to relate $\vec{R}^{>0}_{v, w}$ to $\vec{R}^{>0}_{v', w'}$ and $\overleftarrow{R}^{>0}_{v, w}$ to $\overleftarrow{R}^{>0}_{v', w'}$. We will achieve this by introducing two maps: $g \mapsto g^\theta \dot{w}_0$ and $g \mapsto g^{-\iota} \dot{w}_0$, where $g^{-\iota} := (g^{-1})^\iota = (g^\iota)^{-1}$.

The first map $g \mapsto g^\theta \dot{w}_0$ descends to a map on G/B_-: for $b \in B_-$, it sends $gb \mapsto g^\theta b^\theta \dot{w}_0 = g^\theta \dot{w}_0 \cdot \dot{w}_0^{-1} b^\theta_- \dot{w}_0$, where $b^\theta_- \in B$ and thus $\dot{w}_0^{-1} b^\theta_- \dot{w}_0 \in B_-$. The fact that this map preserves total positivity was recently shown by Lusztig.

Proposition 6.3 ([Lus21, §1.5]). *The map* $\vec{\mathrm{rev}} : G/B_- \xrightarrow{\sim} G/B_-$ *sending* $gB_- \mapsto g^\theta \dot{w}_0 B_-$ *induces homeomorphisms*

$$(G/B_-)_{\geq 0} \xrightarrow{\sim} (G/B_-)_{\geq 0} \quad and \quad \vec{R}^{>0}_{v,w} \xrightarrow{\sim} \vec{R}^{>0}_{v',w'}.$$

The second map $g \mapsto g^{-\iota} \dot{w}_0$ descends to a map on $B_- \backslash G$: for $b \in B_-$, it sends $b_- g \mapsto b_-^{-\iota} g^{-\iota} \dot{w}_0$, where $b_-^{-\iota} \in B_-$. It turns out that this map also preserves total positivity.

Proposition 6.4. *The map* $\vec{\mathrm{rev}} : B_- \backslash G \xrightarrow{\sim} B_- \backslash G$ *sending* $B_- g \mapsto B_- g^{-\iota} \dot{w}_0$ *induces homeomorphisms*

$$(B_- \backslash G)_{\geq 0} \xrightarrow{\sim} (B_- \backslash G)_{\geq 0} \quad and \quad \overleftarrow{R}^{>0}_{v,w} \xrightarrow{\sim} \overleftarrow{R}^{>0}_{v',w'}.$$

Proof. Recall from (3.5) that the map $B_- g \mapsto g^\iota B_-$ yields an involutive isomorphism $(B_- \backslash G)_{\geq 0} \xrightarrow{\sim} (G/B_-)_{\geq 0}$. Letting $h := g^\iota$, we need to show that the map $hB_- \mapsto \dot{w}_0 h^{-\iota} B_-$ induces a homeomorphism $(G/B_-)_{\geq 0} \xrightarrow{\sim} (G/B_-)_{\geq 0}$. We may assume that $h = \mathbf{g}_{\mathbf{v},\mathbf{w}}(\mathbf{t}) = g_1 g_2 \cdots g_m$ is MR-parametrized as in (2.8). Write

$$(6.3) \qquad \dot{w}_0 h^{-\iota} B_- = \dot{w}_0 g_1^{-\iota} \dot{w}_0^{-1} \cdot \dot{w}_0 g_2^{-\iota} \dot{w}_0^{-1} \cdots \dot{w}_0 g_m^{-\iota} \dot{w}_0^{-1} \cdot \dot{w}_0 B_-.$$

If $g_1 = x_i(t)$ then $g_1^{-\iota} = x_i(-t)$ and by Lemma 3.7(v), $\dot{w}_0 g_1^{-\iota} \dot{w}_0^{-1} = y_{i^*}(t)$, which is equal to $x_{i^*}(t)^\theta$. Set

$$(6.4) \qquad \mathbf{g}^*_{\mathbf{v},\mathbf{w}}(\mathbf{t}) := g_1^* g_2^* \cdots g_m^*, \quad where \quad g_r^* = \begin{cases} \dot{s}_{i_r^*}, & \text{if } r \notin J^\circ_{\mathbf{v}}, \\ x_{i^*}(t_r), & \text{if } r \in J^\circ_{\mathbf{v}}. \end{cases}$$

Let $w^* := w_0 w w_0$. For $\mathbf{w} = (i_1, \ldots, i_m)$, let $\mathbf{w}^* := (i_1^*, \ldots, i_m^*)$, and let $\mathbf{v}^*$ be the positive distinguished subexpression for $v^* := w_0 v w_0$ inside $\mathbf{w}^*$. Comparing (6.4) to (2.8), we see that $\mathbf{g}^*_{\mathbf{v},\mathbf{w}}(\mathbf{t}) = \mathbf{g}_{\mathbf{v}^*,\mathbf{w}^*}(\mathbf{t})$, and thus $\mathbf{g}^*_{\mathbf{v},\mathbf{w}}(\mathbf{t}) B_- \in (G/B_-)_{\geq 0}$. In particular, (6.3) implies that

$$\dot{w}_0 h^{-\iota} B_- = \mathbf{g}^*_{\mathbf{v},\mathbf{w}}(\mathbf{t})^\theta \dot{w}_0 B_-.$$

The result belongs to $(G/B_-)_{\geq 0}$ by Proposition 6.3. □

6.3. **The opposite chiral map.** The chiral map relates the opposite Schubert cells $B_- \backslash (B_- vB)$ and $(BvB_-)/B_-$ for v. There is a similar map, relating the Schubert cells $B_- \backslash (B_- wB_-)$ and $(B_- wB_-)/B_-$ for w. First, for $h \in G$, write $h^{-T} := (h^{-1})^T = (h^T)^{-1}$. Recall from (3.13) and (3.14) that we have isomorphisms

$$(6.5) \qquad N_- \dot{w} \cap \dot{w} N \xrightarrow{\sim} (B_- wB_-)/B_-, \qquad\qquad h \mapsto hB_-;$$

$$(6.6) \qquad N_- \dot{w} \cap \dot{w} N \xrightarrow{\sim} B_- \backslash (B_- wB_-), \qquad\qquad h \mapsto B_- h^{-T}.$$

Definition 6.5. Let $w \in W$. The *opposite chiral map*

$$\vec{\chi}^w : B_- \backslash (B_- wB_-) \xrightarrow{\sim} (B_- wB_-)/B_-$$

is obtained by identifying both sides with $N_- \dot{w} \cap \dot{w} N$ via (6.5)–(6.6). We also consider restrictions $\vec{\chi}^w : \overleftarrow{R}^\circ_{v,w} \xrightarrow{\sim} \vec{R}^\circ_{v,w}$ for each $v \leq w$. The inverse of $\vec{\chi}^w$ is denoted $\overleftarrow{\chi}^w$.

We have the following direct analog of Theorem 2.3.

Theorem 6.6. *Let $w \in W$. Then for any $h \in N_- \dot{w} \cap \dot{w} N$, we have*

$$(6.7) \qquad\qquad hB_- \in (G/B_-)_{\geq 0} \quad \Longleftrightarrow \quad B_- h^{-T} \in (B_- \backslash G)_{\geq 0}.$$

In other words, for all $v \leq w \in W$, the map $\vec{\chi}^w : \overleftarrow{R}^\circ_{v,w} \xrightarrow{\sim} \vec{R}^\circ_{v,w}$ restricts to a homeomorphism

$$\vec{\chi}^w : \overleftarrow{R}^{>0}_{v,w} \xrightarrow{\sim} \vec{R}^{>0}_{v,w}.$$

Proof. Similarly to Section 6.2, set $v' := ww_0$ and $w' := vw_0$. We have the following commutative diagram consisting of four involutive isomorphisms:

$$(6.8) \quad
\begin{array}{ccc}
(N\dot{v} \cap \dot{v}N)^{\cap w} & \xrightarrow{\ g_\cap \mapsto g_\cap^\theta \dot{w}_0\ } & N_- \dot{w}' \cap \dot{w}'N \cap B\dot{v}'B_- \\
{\scriptstyle g_\cap \mapsto g_\cap} \downarrow & & \downarrow {\scriptstyle h \mapsto h^{-T}} \\
(N\dot{v} \cap \dot{v}N)^{\cap w} & \xrightarrow{\ g_\cap \mapsto g_\cap^{-\iota} \dot{w}_0\ } & N\dot{w}' \cap \dot{w}'N_- \cap B_- \dot{v}'B.
\end{array}$$

We have already shown that the left vertical map and both horizontal maps preserve total positivity. More precisely, fix $g_\cap \in (N\dot{v} \cap \dot{v}N)^{\cap w}$. By Proposition 6.3, $g_\cap B_- \in (G/B_-)_{\geq 0}$ if and only if $g_\cap^\theta \dot{w}_0 B_- \in (G/B_-)_{\geq 0}$. By Theorem 2.3, $g_\cap B_- \in (G/B_-)_{\geq 0}$ if and only if $B_- g_\cap \in (B_- \backslash G)_{\geq 0}$. By Proposition 6.4, $B_- g_\cap \in (B_- \backslash G)_{\geq 0}$ if and only if $B_- g_\cap^{-\iota} \dot{w}_0 \in (B_- \backslash G)_{\geq 0}$. Setting $h := g_\cap^\theta \dot{w}_0$ and composing the three maps, we see that $h \in (G/B_-)_{\geq 0}$ if and only if $h^{-T} B_- \in (G/B_-)_{\geq 0}$. Thus indeed the map $\vec{\chi}^w$ preserves total positivity, i.e., (6.7) is satisfied for all $h \in N_- \dot{w} \cap \dot{w} N$. $\square$

7. Twist maps and total positivity

In this section, we show that the twist and pre-twist maps preserve total positivity. We fix a pair $v \leq w$ in W, an MR-parametrized element $g = \mathbf{g}_{\mathbf{v},\mathbf{w}}(\mathbf{t}) \in (N\dot{v})^{\cap w}$ with $\mathbf{t} \in (\mathbb{R}_{>0})^{J^\circ_{\mathbf{v}}}$, and the corresponding element $g_\cap \in (N\dot{v} \cap \dot{v}N)^{\cap w}$ satisfying $gB_- = g_\cap B_-$. Recall that Lemma 3.4 gives an explicit procedure to recover $g_\cap$ from g.

In view of Lemma 3.4 and (5.3), we consider factorizations

$$(7.1) \qquad\qquad g^T \dot{w} = y_+ y_0 y_- \quad \text{and} \quad g_\cap^T \dot{w} = y'_+ y_0 y_-$$

for $(y_+, y_0, y_-) \in N \times H \times N_-$ and $y'_+ = x^T y_+ \in N$, where x is given by Lemma 3.4. See (2.17) for an example. By definition, we have

$$(7.2) \qquad\qquad \vec{y} = [g^T \dot{w}]^+_L = y_+.$$

Our first goal is to express the element y_0 in terms of the MR-parameters $\mathbf{t} \in (\mathbb{R}_{>0})^{J^\circ_{\mathbf{v}}}$.

Lemma 7.1. *The element $y_0 \in H$ is given by*

$$y_0 = \prod_{j \in J^\circ_{\mathbf{v}}} t_j^{-s_{i_m} \cdots s_{i_{j+1}} \alpha^\vee_{i_j}}.$$

In particular, we have $y_0 \in H_{>0}$ and for each $i \in I$,

$$(7.3) \qquad\qquad \Delta^{\omega_i}_{\omega_i}(y_0) = \prod_{j \in J^\circ_{\mathbf{v}}} t_j^{-\langle \omega_i, s_{i_m} \cdots s_{i_{j+1}} \alpha^\vee_{i_j} \rangle}.$$

Proof. We have $y_0 = [g^T \dot{w}]_0^{\pm} = [g_m^T \cdots g_1^T \dot{s}_{i_1} \cdots \dot{s}_{i_m}]_0^{\pm}$. We prove the following stronger statement by induction on $\ell(w)$: for any $h \in N$, we have

(7.4)
$$[g^T h \dot{w}]_0^{\pm} = \prod_{j \in J_{\mathbf{v}}^{\circ}} t_j^{-s_{i_m} \cdots s_{i_{j+1}} \alpha_{i_j}^{\vee}}.$$

In the base case $w = \mathrm{id}$, both sides of (7.4) are equal to the identity element of H. Assume now that the result is known for all $w' \in W$ with $\ell(w') < \ell(w)$. We consider two cases: either $g_1^T = \dot{s}_i^{-1}$ or $g_1^T = y_i(t)$ for some $i \in I$ and $t > 0$. In each case, we set $w' := s_i w < w$ and $g' := g_2 \cdots g_m$.

In the first case $g_1^T = \dot{s}_i^{-1}$, we have

$$[g^T h \dot{w}]_0^{\pm} = [(g')^T \dot{s}_i^{-1} h \dot{s}_i \dot{w}']_0^{\pm}.$$

By Lemma 3.7(iv), we can factorize $\dot{s}_i^{-1} h \dot{s}_i = h' y_i(t')$ for $h' \in N$ and $t' \in \mathbb{R}$. By Lemma 3.7(ii), $y_i(t') \dot{w}' \in \dot{w}' N_-$, and thus we get $[g^T h \dot{w}]_0^{\pm} = [(g')^T h' \dot{w}']_0^{\pm}$. The result follows from the induction hypothesis.

Consider the second case $g_1^T = y_i(t)$. Using Lemma 3.2, we factorize $h = h_1 h_2$ for $(h_1, h_2) \in N'(s_i) \times N(s_i)$. Thus $h_2 = x_i(t')$ for some $t' \in \mathbb{C}$. Since $s_i w < w$, by Lemma 3.7(i), we get $h_2 \dot{w} \in \dot{w} N_-$, and thus $[g^T h \dot{w}]_0^{\pm} = [g^T h_1 \dot{w}]_0^{\pm}$. Write

$$[g^T h_1 \dot{w}]_0^{\pm} = [(g')^T y_i(t) h_1 \dot{s}_i \dot{w}']_0^{\pm} = [(g')^T \cdot y_i(t) h_1 y_i(-t) \cdot y_i(t) \dot{s}_i \dot{w}']_0^{\pm}.$$

Applying a collision move (3.18), we write $y_i(t) \dot{s}_i = x_i(t_+) a_+^{-1} y_i(t_-)$ where $t_+ = 1/t$ and $a_+ = t^{\alpha_i^{\vee}}$. By Lemma 3.7(ii), $y_i(t_-) \dot{w}' \in \dot{w}' N_-$. Next, we have $t^{-\alpha_i^{\vee}} \dot{w}' = \dot{w}' t^{-s_{i_m} \cdots s_{i_2} \alpha_i^{\vee}}$. By Lemma 3.7(iii), we have $y_i(t) h_1 y_i(-t) \in N$. Setting $h' := y_i(t) h_1 y_i(-t) x_i(t_+)$, we get

$$[g^T h_1 \dot{w}]_0^{\pm} = [(g')^T h' \dot{w}']_0^{\pm} \cdot t^{-s_{i_m} \cdots s_{i_2} \alpha_i^{\vee}}.$$

This finishes the induction step. $\qquad\square$

We now use the above computation to show that the twist maps preserve total positivity.

Proof of Theorem 2.6. In view of Theorem 2.3, it suffices to show that the right pre-twist map $\vec{\tau}_{v,w}^{\mathrm{pre}} : \vec{R}_{v,w}^{\circ} \xrightarrow{\sim} \check{R}_{v,w}^{\circ}$ sends $\vec{R}_{v,w}^{>0}$ to $\check{R}_{v,w}^{>0}$. Recall from (5.3) that the element y_- belongs to $\dot{w}^{-1} N \dot{w} \cap N_-$. Let $h := \dot{w} y_-^T$, and thus $h \in N_- \dot{w} \cap \dot{w} N$. By (5.4), we have $Bg^T = Bh^T$, and thus $gB_- = hB_-$ belongs to $(G/B_-)_{\geq 0}$. By Theorem 6.6, we get

$$B_- h^{-T} = B_- \dot{w} y_-^{-1} \quad \in (B_- \backslash G)_{\geq 0}.$$

Since $y_0 \in H_{>0}$ by Lemma 7.1, we can act by y_0^{-1} on the right and get $B_- \dot{w} y_-^{-1} y_0^{-1} \in (B_- \backslash G)_{\geq 0}$. By (7.1), $y_0 y_- = y_+^{-1} g^T \dot{w}$, so $y_-^{-1} y_0^{-1} = \dot{w}^{-1} g^{-T} y_+$. We have shown that $B_- g^{-T} y_+ \in (B_- \backslash G)_{\geq 0}$. Since $g \in N \dot{v}$, we have $g^{-T} \in N_- \dot{v}$, so $B_- g^{-T} y_+ = B_- \dot{v} y_+ \in (B_- \backslash G)_{\geq 0}$. It remains to note that the right pre-twist of gB_- is given by

$$\vec{\tau}_{v,w}^{\mathrm{pre}}(gB_-) := B_- \dot{v} [g^T \dot{w}]_L^+ = B_- \dot{v} y_+ \quad \in (B_- \backslash G)_{\geq 0}. \qquad\square$$

8. Right twist Chamber Ansatz

We concentrate on proving Theorem 2.8. Even though it looks similar to the Marsh–Rietsch Chamber Ansatz (Theorem 5.1), we were unable to formally deduce one statement from the other. Thus, we utilize the proof strategy of [BZ97] who treated the case $v = \mathrm{id}$.

Throughout, let us fix a pair $v \leq w$ in W, a reduced expression $\mathbf{w}$ for w, and an MR-parametrized element $g = \mathbf{g}_{\mathbf{v},\mathbf{w}}(\mathbf{t}) = g_1 g_2 \cdots g_m$ with $\mathbf{t} \in (\mathbb{C}^*)^{J_{\mathbf{v}}^{\circ}}$. Our goal is to study the

element $\vec{y} := [g^T \dot{w}]_L^+$. Recall from (3.12) that for $u \in W$, we set $N(u) := N \cap \dot{u}^{-1} N_- \dot{u}$ and $N'(u) := N \cap \dot{u}^{-1} N \dot{u}$.

Lemma 8.1. *Let* $1 \le r \le m+1$ *and*

$$(8.1) \qquad \vec{y}^{(r)} := [g_m^T \cdots g_r^T \dot{s}_{i_r} \cdots \dot{s}_{i_m}]_L^+.$$

Then $\vec{y} \in \vec{y}^{(r)} \cdot N'(s_{i_r} \cdots s_{i_m})$.

Proof. We first show that for all $h \in N$, we have

$$(8.2) \qquad [g^T h \dot{w}]_L^+ \in \vec{y} \cdot N'(w).$$

By Lemma 3.2, we can factorize $h = h_1 h_2$ for $(h_1, h_2) \in N'(w^{-1}) \times N(w^{-1})$. Since $h_2 \dot{w} \in \dot{w} N_-$, we have $[g^T h \dot{w}]_L^+ = [g^T h_1 \dot{w}]_L^+$. Next, by (7.1), we have

$$g^T h_1 \dot{w} = g^T \dot{w} \cdot \dot{w}^{-1} h_1 \dot{w} = y_+ y_0 y_- \cdot \dot{w}^{-1} h_1 \dot{w}.$$

By (5.3), $y_- \in N_- \cap \dot{w}^{-1} N \dot{w}$. Note that $\dot{w}^{-1} h_1 \dot{w} \in N'(w)$. By Lemma 3.2, the multiplication map gives isomorphisms

$$(N_- \cap \dot{w}^{-1} N \dot{w}) \times N'(w) \to \dot{w}^{-1} N \dot{w} \to N'(w) \times (N_- \cap \dot{w}^{-1} N \dot{w}).$$

Thus

$$g^T h_1 \dot{w} \in y_+ y_0 \cdot N'(w) \cdot (N_- \cap \dot{w}^{-1} N \dot{w}) \subset y_+ \cdot N'(w) \cdot H \cdot (N_- \cap \dot{w}^{-1} N \dot{w}).$$

We therefore find

$$[g^T h \dot{w}]_L^+ = [g^T h_1 \dot{w}]_L^+ \in y_+ \cdot N'(w).$$

It remains to note that by (7.2), $y_+ = \vec{y}$.

Clearly, (8.2) holds with $(w, g, \vec{y})$ replaced with $(s_{i_r} \cdots s_{i_m}, g_r \cdots g_m, \vec{y}^{(r)})$: for all $h \in N$, we have

$$(8.3) \qquad [g_m^T \cdots g_r^T h \dot{s}_{i_r} \cdots \dot{s}_{i_m}]_L^+ \in \vec{y}^{(r)} \cdot N'(s_{i_r} \cdots s_{i_m}).$$

Finally, we show that for all $h^{(1)} \in N$ and all $1 \le r \le m+1$, there exists $h^{(r)} \in N$ such that

$$(8.4) \qquad [g^T h^{(1)} \dot{w}]_L^+ = [g_m^T \cdots g_r^T h^{(r)} \dot{s}_r \cdots \dot{s}_m]_L^+.$$

We proceed by induction on r. The base case $r = 1$ is clear. To pass from $r = 1$ to $r = 2$, we consider two cases: either $g_1^T = \dot{s}_i^{-1}$ or $g_1^T = y_i(t)$ for some $i \in I$ and $t > 0$. In each case, we set $w' := s_i w < w$ and $g' := g_2 \cdots g_m$. The proof of Lemma 7.1 shows that in each case, we have $[g^T h^{(1)} \dot{w}]_L^+ = [(g')^T h^{(2)} \dot{w}']_L^+$ for some $h^{(2)} \in N$. Iterating this procedure, we prove (8.4) for all $1 \le r \le m+1$.

To finish the proof of the lemma, we observe that $\vec{y} = [g^T h^{(1)} \dot{w}]_L^+$ for $h^{(1)} = 1 \in N$. By (8.4), $\vec{y} = [g_m^T \cdots g_r^T h^{(r)} \dot{s}_r \cdots \dot{s}_m]_L^+$ for some $h^{(r)} \in N$. Applying (8.3), we get $\vec{y} \in \vec{y}^{(r)} \cdot N'(s_{i_r} \cdots s_{i_m})$. $\square$

Corollary 8.2. *For each* $1 \le r \le m$, *we have*

$$\vec{f}_r(\vec{y}) = \vec{f}_r(\vec{y}^{(r)}).$$

Proof. Recall from (2.9) that $\vec{f}_r = \Delta_{w^{(r-1)} \omega_{i_r}}^{v^{(r-1)} \omega_{i_r}} = \Delta_{s_{i_m} \cdots s_{i_r} \omega_{i_r}}^{s_{i_m}^{\mathbf{v}} \cdots s_{i_r}^{\mathbf{v}} \omega_{i_r}}$. Applying Lemma 3.3 to the pair $(s_{i_r}^{\mathbf{v}} \cdots s_{i_m}^{\mathbf{v}} \le s_{i_r} \cdots s_{i_m})$, we see that the function $\vec{f}_r \in \mathbb{C}[N]$ is invariant under right multiplication by $N'(s_{i_r} \cdots s_{i_m})$. The result follows from Lemma 8.1. $\square$

Lemma 8.3. *For all $i \in I$, we have*

$$\Delta^{\omega_i}_{w^{-1}\omega_i}(g) = \prod_{r \in J^\circ_{\mathbf{v}}} t_r^{\langle \omega_i, s_{i_1} \cdots s_{i_{r-1}} \alpha^\vee_{i_r} \rangle}$$

Proof. We have

$$\Delta^{\omega_i}_{w^{-1}\omega_i}(g) = \Delta^{\omega_i}_{\omega_i}(g\dot{w}^{-1}) = \Delta^{\omega_i}_{\omega_i}([g\dot{w}^{-1}]^{\mp}_0).$$

The computation of $\Delta^{\omega_i}_{\omega_i}([g\dot{w}^{-1}]^{\mp}_0)$ is entirely analogous to the computation of $\Delta^{\omega_i}_{\omega_i}(y_0) = \Delta^{\omega_i}_{\omega_i}([g^T \dot{w}]^{\pm}_0)$ which was completed in Lemma 7.1. □

Lemma 8.4. *For all $i \in I$, we have*

$$\Delta^{v^{-1}\omega_i}_{w^{-1}\omega_i}(\vec{y}) \cdot \Delta^{\omega_i}_{w^{-1}\omega_i}(g) = 1.$$

Proof. Let $\hat{y} := \dot{v}\vec{y}$. By Proposition 2.4, we have $\hat{y} \in (\dot{v}N)^{\cap w}$. By (3.16), we have $\dot{w}\hat{y}^T \in G^{\mp}_0$. Let $(x_-, x_0, x_+) \in N_- \times H \times N$ be given by

$$(8.5) \qquad\qquad \dot{w}\vec{y}^T\dot{v}^{-1} = \dot{w}\hat{y}^T = x_- x_0 x_+.$$

Our first goal is to show that $x_+ = g\dot{v}^{-1}$. Since $x_+ \in N$ and $g \in N\dot{v}$, it suffices to show that $B_- x_+ = B_- g\dot{v}^{-1}$. Recall from (7.1) that we have a factorization $g^T \dot{w} = y_+ y_0 y_-$, and as explained in (5.3), we have $y_- \in \dot{w}^{-1}N\dot{w} \cap N_-$. In particular, $y^T_- \in \dot{w}^{-1}N_-\dot{w}$, and thus $\dot{w}y^T_- y^T_0 \dot{w}^{-1} \in B_-$. By (7.2), we have $\vec{y} = y_+$. Applying (8.5), we find

$$B_- x_+ = B_- x_- x_0 x_+ = B_- \dot{w}\vec{y}^T \dot{v}^{-1} = B_- \dot{w}y^T_- y^T_0 \dot{w}^{-1} \cdot \dot{w}\vec{y}^T \dot{v}^{-1} = B_- \dot{w} \cdot \dot{w}^{-1}g\dot{v}^{-1} = B_- g\dot{v}^{-1}.$$

Thus indeed $x_+ = g\dot{v}^{-1}$.

By Lemma 3.2, $\dot{w}\vec{y}^T\dot{w}^{-1} \in N_- N$. By (3.8)–(3.10), we have

$$1 = \Delta^{\omega_i}_{\omega_i}(\dot{w}\vec{y}^T\dot{w}^{-1}) = \Delta^{\omega_i}_{\omega_i}(\dot{w}(\dot{w}^{-1}x_- x_0 x_+ \dot{v})\dot{w}^{-1}) = \Delta^{\omega_i}_{\omega_i}(x_0 x_+ \dot{v}\dot{w}^{-1}) = \Delta^{\omega_i}_{\omega_i}(x_0)\Delta^{\omega_i}_{\omega_i}(x_+ \dot{v}\dot{w}^{-1})$$

$$= \Delta^{\omega_i}_{\omega_i}(x_0)\Delta^{\omega_i}_{\omega_i}(g\dot{w}^{-1}) = \Delta^{\omega_i}_{\omega_i}(x_0)\Delta^{\omega_i}_{w^{-1}\omega_i}(g).$$

It remains to show that $\Delta^{\omega_i}_{\omega_i}(x_0) = \Delta^{v^{-1}\omega_i}_{w^{-1}\omega_i}(\vec{y})$. From (8.5), we find $\vec{y} = \dot{v}^{-1}x^T_+ x_0 x^T_- \dot{w}$, and thus

$$\Delta^{v^{-1}\omega_i}_{w^{-1}\omega_i}(\vec{y}) = \Delta^{\omega_i}_{\omega_i}(\dot{v}(\dot{v}^{-1}x^T_+ x_0 x^T_- \dot{w})\dot{w}^{-1}) = \Delta^{\omega_i}_{\omega_i}(x^T_+ x_0 x^T_-) = \Delta^{\omega_i}_{\omega_i}(x_0),$$

since $x^T_+ \in N_-$ and $x^T_- \in N$. □

Proof of Theorem 2.8. We first show (2.11). By Corollary 8.2, we have $\vec{f}_j(\vec{y}) = \vec{f}_j(y^{(j)})$. Let $g^{(j)} := g_j \cdots g_m$ and let $\vec{y}^{(j)}$ be given by (8.1). By Lemma 8.4, we have

$$\vec{f}_j(\vec{y}^{(j)}) \cdot \Delta^{\omega_{i_j}}_{s_{i_m}\cdots s_{i_j}\omega_{i_j}}(g^{(j)}) = 1.$$

By Lemma 8.3,

$$\Delta^{\omega_{i_j}}_{s_{i_m}\cdots s_{i_j}\omega_{i_j}}(g^{(j)}) = \prod_{r \in J^\circ_{\mathbf{v}} : r \geq j} t_r^{\langle \omega_{i_j}, s_{i_j} s_{i_{j+1}} \cdots s_{i_{r-1}} \alpha^\vee_{i_r} \rangle}.$$

This shows (2.11).

Let us now deduce (2.10) from (2.11). Each side of (2.10) is a monomial in $(t_k)_{k \geq r}$. Let $k \in J^\circ_{\mathbf{v}}$ be such that $k \geq r$. We will show that t_k appears on both sides of (2.10) with the same exponent. Assume first that $k > r$. By (2.11), the exponent of t_k in the numerator of the right-hand side of (2.10) is given by

$$\langle \sum_{j \neq i_r} a_{j,i_r}\omega_j, s_{i_{r+1}} \cdots s_{i_{k-1}} \alpha^\vee_{i_k} \rangle.$$

On the other hand, the exponent of t_k in the denominator is

$$\langle -\omega_{i_r}, s_{i_r} \cdots s_{i_{k-1}} \alpha_{i_k}^\vee \rangle + \langle -\omega_{i_r}, s_{i_{r+1}} \cdots s_{i_{k-1}} \alpha_{i_k}^\vee \rangle.$$

Since $s_{i_r} \omega_{i_r} = \omega_{i_r} - \alpha_{i_r}$, we find

$$\langle -\omega_{i_r}, s_{i_r} \cdots s_{i_{k-1}} \alpha_{i_k}^\vee \rangle = \langle \alpha_{i_r} - \omega_{i_r}, s_{i_{r+1}} \cdots s_{i_{k-1}} \alpha_{i_k}^\vee \rangle.$$

Therefore the total exponent of t_k in the right-hand side of (2.10) is

$$\langle -\alpha_{i_r} + 2\omega_{i_r} + \sum_{j \neq i_r} a_{j,i_r} \omega_j, s_{i_{r+1}} \cdots s_{i_{k-1}} \alpha_{i_k}^\vee \rangle = \langle -\alpha_{i_r} + \sum_{j \in I} a_{j,i_r} \omega_j, s_{i_{r+1}} \cdots s_{i_{k-1}} \alpha_{i_k}^\vee \rangle = 0,$$

since $\alpha_{i_r} = \sum_{j \in I} a_{j,i_r} \omega_j$. We have matched the exponent of t_k on both sides of (2.10) for all $k > r$. For $k = r$, by (2.11), t_r appears in $\vec{f}_j(\vec{y})$ only for $j \leq r$. Thus, out of all terms in the right-hand side of (2.10), it only contributes to $\vec{f}_r(\vec{y}) = \Delta^{v^{(r-1)}\omega_{i_r}}_{w^{(r-1)}\omega_{i_r}}(\vec{y})$ in the denominator. It is clear from (2.11) that the exponent of t_r in $\vec{f}_r(\vec{y})$ is equal to -1. $\qquad\square$

Comparing Theorems 2.8 and 5.1 with Lemma 7.1, we obtain the following relation between the left and the right twist Chamber Ansatz formulas. Set

$$(8.6) \qquad\qquad z := (\dot{w} y_- \dot{w}^{-1})^T.$$

Thus $z \in N_- \cap \dot{w} N \dot{w}^{-1}$. By (5.4), we have $Bg^T = B\dot{w}^{-1} z^T$, and thus $gB_- = z\dot{w}B_-$ belongs to $(G/B_-)_{\geq 0}$. We conclude that the element z satisfies the assumptions of Theorem 5.1. Theorem 2.8 is stated in terms of the element $\vec{y} := [g^T \dot{w}]_L^+$, which by (7.2) equals y_+.

Corollary 8.5. *For all $i \in I$ and $r = 1, \ldots, m+1$, we have*

$$(8.7) \qquad\qquad \Delta^{s_{i_m}^\vee \cdots s_{i_r}^\vee \omega_i}_{s_{i_m} \cdots s_{i_r} \omega_i}(y_+ y_0) = \Delta^{s_1 \cdots s_{i_{r-1}} w_0 \omega_{i^*}}_{s_1^\vee \cdots s_{i_{r-1}}^\vee w_0 \omega_{i^*}}(z^T).$$

Remark 8.6. Corollary 8.5 demonstrates the primary difficulty with relating the two Chamber Ansatz formulas: Theorem 2.8 is given in terms of $\vec{y} = y_+$, while Theorem 2.12 is given in terms of $\bar{y}$ which is closely related to y_-, as explained in the proof of Theorem 2.12. It is unclear to us how to show (8.7) directly without reproving each Chamber Ansatz from scratch.

Remark 8.7. Let $G = \mathrm{SL}_n(\mathbb{C})$. Let C be a chamber of the wiring diagram of $\mathbf{w}$ located vertically at height i and horizontally between the $(r-1)$-th and the r-th crossings. Then the left- and the right-hand sides of (8.7) are equal to $\vec{\Delta}_C(y_+ y_0)$ and $\bar{\Delta}_C(z^T)$, respectively.

Example 8.8. We continue the example from Section 2.9. Recall that the matrices y_+, y_0, y_- defined in (7.1) have been computed in the right-hand side of (2.17). Thus

$$y_+ = \begin{pmatrix} 1 & \frac{1}{t_6} & \frac{t_3}{t_1} & 0 \\ 0 & 1 & \frac{t_3 t_6}{t_1} & -\frac{1}{t_4} \\ 0 & 0 & 1 & \frac{1}{t_3 t_4} \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad z^T = \dot{w} y_- \dot{w}^{-1} = \begin{pmatrix} 1 & \frac{t_1}{t_3} & \frac{1}{t_3} & \frac{1}{t_3 t_4} \\ 0 & 1 & \frac{1}{t_1} & \frac{1}{t_1 t_4} \\ 0 & 0 & 1 & \frac{t_1+t_6}{t_4 t_6} \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The minors of these matrices appearing in Corollary 8.5 are computed in Figure 2.

9. Muller–Speyer twist
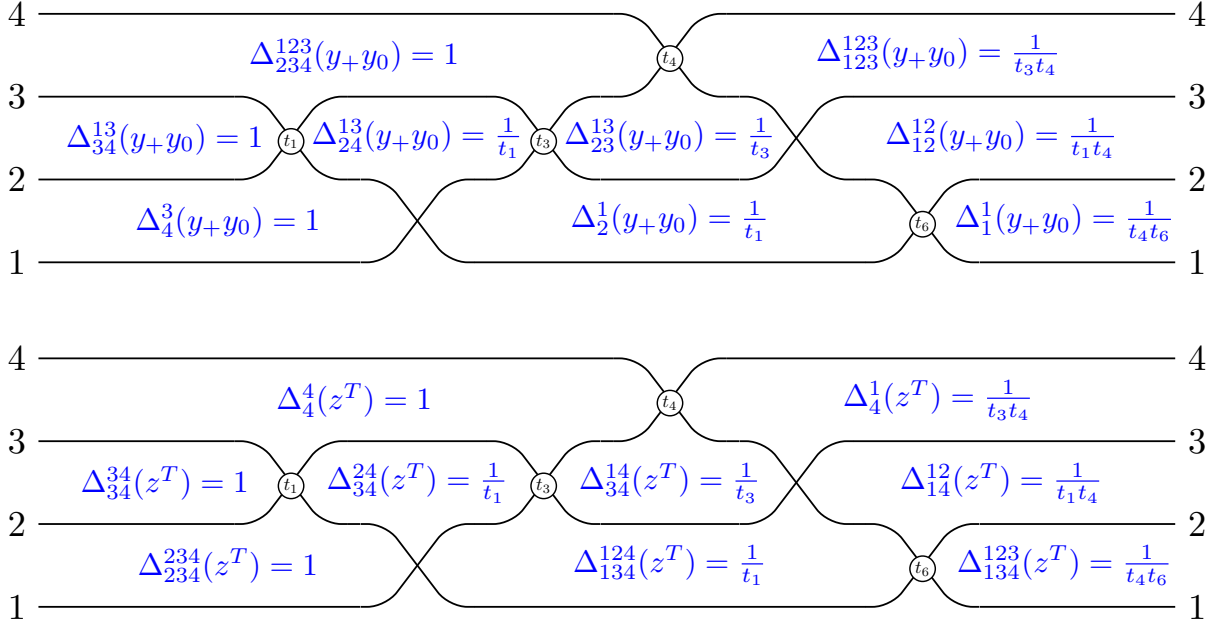
We relate our twist maps to the twist maps of [MuSp17].

FIGURE 2. Relating the left and the right twist Chamber Ansatz formulas; cf. Corollary 8.5 and Example 8.8.

9.1. Background on positroid varieties.

We continue using the notation from Section 2.5. Recall that for $g \in G$, $\pi_J(gB_-) \in \mathrm{Gr}(n-k, n)$ is the linear span of the rightmost $n - k$ columns of the matrix g. We denote by $|g|$ the corresponding $n \times (n - k)$ submatrix of g with column set $[k + 1, n] := \{k + 1, k + 2, \dots, n\}$. For an $(n - k)$-element set $F \subset [n]$ and $g \in G$, we let $\Delta_F(g)$ be the minor of g with row set F and column set $[k + 1, n]$. The functions $\Delta_F(g)$ give projective coordinates on $\mathrm{Gr}(n-k, n)$ called *Plücker coordinates*.

Recall that for $(v, w) \in Q^J$, we have the open positroid variety $\Pi^\circ_{v,w} = \pi_J(\vec{R}^\circ_{v,w})$. The *positroid cell* $\Pi^{>0}_{v,w}$ is the image of $\vec{R}^{>0}_{v,w}$ under π_J. Alternatively, it can be described as the subset of $\Pi^\circ_{v,w}$ where all nonzero Plücker coordinates have the same sign.

9.2. The Muller–Speyer twist map.

Fix $(v, w) \in Q^J$. We define the *right twist* $\vec{\tau}^{\mathrm{MS}}_{v,w} : \Pi^\circ_{v,w} \xrightarrow{\sim} \Pi^\circ_{v,w}$. Let $g \in G$ be such that $gP^J_- \in \Pi^\circ_{v,w}$, and let $M := |g|$ be the corresponding $n \times (n-k)$ matrix. Thus gP^J_- is the column span of M. We denote by $M_1, M_2, \dots, M_n \in \mathbb{C}^{n-k}$ the row vectors of M. We extend this labeling periodically for all $i \in \mathbb{Z}$ by the condition that $M_{i+n} = M_i$ for all $i \in \mathbb{Z}$. Let $\langle \cdot, \cdot \rangle$ denote the standard inner product on $\mathbb{C}^{n-k}$. For $i \in \mathbb{Z}$, let

$$\vec{I}_i := \{j \geq i \mid M_j \text{ is not in the span of } M_i, M_{i+1}, \dots, M_{j-1}\};$$
$$\overleftarrow{I}_i := \{j \leq i \mid M_j \text{ is not in the span of } M_i, M_{i-1}, \dots, M_{j+1}\}.$$

By definition, the span of the empty set is $\{0\}$, and thus when $M_i = 0$, we have $i \notin \vec{I}_i$ and $i \notin \overleftarrow{I}_i$. We always have $|\vec{I}_i| = |\overleftarrow{I}_i| = n - k$ and each of the sets $\{M_j \mid j \in \vec{I}_i\}$ and $\{M_j \mid j \in \overleftarrow{I}_i\}$ forms a basis of $\mathbb{C}^{n-k}$.

For $i \in \mathbb{Z}$, let $\vec{M}_i \in \mathbb{C}^{n-k}$ be the unique vector given by

$$\langle \vec{M}_i, M_j \rangle = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{otherwise,} \end{cases} \quad \text{for all } j \in \vec{I}_i.$$

Similarly, let $\overleftarrow{M}_i \in \mathbb{C}^{n-k}$ be the unique vector given by

$$\langle \overleftarrow{M}_i, M_j \rangle = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{otherwise,} \end{cases} \quad \text{for all } j \in \overleftarrow{I}_i.$$

Let $\vec{M}$ be the matrix with rows $\vec{M}_1, \vec{M}_2, \ldots, \vec{M}_n$, and let $\overleftarrow{M}$ be the matrix with rows $\overleftarrow{M}_1, \overleftarrow{M}_2, \ldots, \overleftarrow{M}_n$. By definition, $\vec{\tau}_{v,w}^{\mathrm{MS}}(gP_-^J)$ is the column span of $\vec{M}$ and $\overleftarrow{\tau}_{v,w}^{\mathrm{MS}}(gP_-^J)$ is the column span of $\overleftarrow{M}$. See Example 9.2 below.

9.3. **Relating the twist maps.** The following result, giving the relationship between our twist maps $\vec{\tau}_{v,w}, \overleftarrow{\tau}_{v,w}$ and the twist maps $\vec{\tau}_{v,w}^{\mathrm{MS}}, \overleftarrow{\tau}_{v,w}^{\mathrm{MS}}$ of [MuSp17], is a reformulation of Theorem 2.7.

Theorem 9.1. *For all $(v, w) \in Q^J$, we have the following commutative diagram, where all maps are isomorphisms:*

(9.1)
$$
\begin{array}{ccccc}
\vec{R}_{v,w}^\circ & \xleftarrow{\overleftarrow{\tau}_{v,w}} & \vec{R}_{v,w}^\circ & \xrightarrow{\vec{\tau}_{v,w}} & \vec{R}_{v,w}^\circ \\
\downarrow{\scriptstyle \pi_J} & & \downarrow{\scriptstyle \pi_J} & & \downarrow{\scriptstyle \pi_J} \\
\Pi_{v,w}^\circ & \xleftarrow[\overleftarrow{\tau}_{v,w}^{\mathrm{MS}}]{} & \Pi_{v,w}^\circ & \xrightarrow[\vec{\tau}_{v,w}^{\mathrm{MS}}]{} & \Pi_{v,w}^\circ.
\end{array}
$$

Proof. We focus on the commutativity of the right square in (9.1). First, we claim that the following diagram is commutative, where all maps are homeomorphisms:

(9.2)

Here $\mathrm{Meas} : (\mathbb{R}_{>0})^{J_{\mathbf{v}}^\circ} \xrightarrow{\sim} \Pi_{v,w}^{>0}$ is the *boundary measurement map* of [Pos06], which puts the positive parameters $(t_r)_{r \in J_{\mathbf{v}}^\circ}$ on the edges of a *Le-diagram plabic graph*; see part (c) in the proof of [GL19, Proposition 4.3]. We have a collection $\{F_0\} \sqcup \{F_r \mid r \in J_{\mathbf{v}}^\circ\}$ of $(n-k)$-element sets labeling the faces of the Le-diagram as in [GL19, Section 1.3]. The corresponding Plücker coordinates are functions on $\Pi_{v,w}^\circ$ defined up to common rescaling, and we always rescale them so that $\Delta_{F_0} = 1$. Thus we obtain a map $(\Delta_{F_r})_{r \in J_{\mathbf{v}}^\circ} : \Pi_{v,w}^{>0} \to (\mathbb{R}_{>0})^{J_{\mathbf{v}}^\circ}$, which appears as the bottom right diagonal arrow in (9.2).

The commutativity of the top trapezoid in (9.2) follows from the right twist Chamber Ansatz (Theorem 2.8). The bottom trapezoid is commutative by the results of [MuSp17]. The left triangle is commutative by the results of [TW13, Kar16]; see the proof of [GL19, Proposition 4.3]. Finally, the right triangle is commutative by [GL19, Lemma 3.6]. We have

shown that the right square in (9.1) is commutative when all maps are restricted to $\vec{R}_{v,w}^{>0}$ and $\Pi_{v,w}^{>0}$. Since these maps are regular and the subsets $\vec{R}_{v,w}^{>0} \subset \vec{R}_{v,w}^{\circ}$ and $\Pi_{v,w}^{>0} \subset \Pi_{v,w}^{\circ}$ are Zariski dense, it follows that the right square in (9.1) is commutative. Since $\overleftarrow{\tau}_{v,w}$ and $\overleftarrow{\tau}_{v,w}^{\mathrm{MS}}$ are inverses of $\vec{\tau}_{v,w}$ and $\vec{\tau}_{v,w}^{\mathrm{MS}}$, respectively, we see that the left square in (9.1) is commutative as well. $\qquad\square$

Example 9.2. Let $k = 2$, $n = 5$, $v = s_4 s_2$, and $\mathbf{w} = (2, 1, 4, 3, 2)$. (Thus $w = s_2 s_1 s_4 s_3 s_2$ is k-Grassmannian since $w s_j > w$ for all $j \neq k$.) We have

$$g = \begin{pmatrix} 1 & 0 & t_2 & 0 & 0 \\ 0 & -t_1 & 1 & t_1 t_4 & 0 \\ 0 & -1 & 0 & t_4 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 \end{pmatrix}, \quad \vec{\tau}_{v,w}(gB_-) = \begin{pmatrix} 1 & \frac{1}{t_1 t_2} & \frac{1}{t_2} & 0 & 0 \\ 0 & 0 & 1 & \frac{1}{t_1 t_4} & 0 \\ 0 & -1 & 0 & \frac{1}{t_4} & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 \end{pmatrix} \cdot B_-.$$

Applying the map $x \mapsto |x|$, we get

$$\pi_J(gB_-) = \mathrm{ColumnSpan} \begin{pmatrix} t_2 & 0 & 0 \\ 1 & t_1 t_4 & 0 \\ 0 & t_4 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \quad \text{and} \quad \pi_J \circ \vec{\tau}_{v,w}(gB_-) = \mathrm{ColumnSpan} \begin{pmatrix} \frac{1}{t_2} & 0 & 0 \\ 1 & \frac{1}{t_1 t_4} & 0 \\ 0 & \frac{1}{t_4} & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}.$$

On the other hand, applying the definition in Section 9.2 to the matrix $M = |g|$, we find

$$\vec{\tau}_{v,w}^{\mathrm{MS}} \circ \pi_J(gB_-) = \mathrm{ColumnSpan}\, \vec{M}, \quad \text{where} \quad \vec{M} = \begin{pmatrix} \frac{1}{t_2} & -\frac{1}{t_1 t_2 t_4} & 0 \\ 1 & 0 & 0 \\ 0 & \frac{1}{t_4} & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$$

Applying column operations, we see that $\pi_J \circ \vec{\tau}_{v,w}(gB_-) = \vec{\tau}_{v,w}^{\mathrm{MS}} \circ \pi_J(gB_-)$ inside $\mathrm{Gr}(n-k, n)$, confirming the commutativity of the right square in (9.1).

10. Subtraction-free expressions

In the previous sections, we have shown that several maps preserve total positivity. In this section, we show the stronger statement that these maps are given by *subtraction-free rational functions* when written in terms of the MR-parameters. In order to state our results precisely, we review the results of [BH21].

Let $\mathbf{k}$ denote a field. We let $G(\mathbf{k}), B(\mathbf{k}), B_-(\mathbf{k}), (G/B_-)(\mathbf{k}), \vec{R}_{v,w}^{\circ}(\mathbf{k})$, and so on denote the respective groups or varieties defined over $\mathbf{k}$. The chiral maps $\vec{\chi}_v, \overleftarrow{\chi}_v$, the opposite chiral maps $\vec{\chi}^w, \overleftarrow{\chi}^w$, the pre-twists $\vec{\tau}_{v,w}^{\mathrm{pre}}, \overleftarrow{\tau}_{v,w}^{\mathrm{pre}}$, the twists $\vec{\tau}_{v,w}, \overleftarrow{\tau}_{v,w}$, and the reversal maps $\vec{\mathrm{rev}}, \overleftarrow{\mathrm{rev}}$ of Section 6.2 are all defined over $\mathbf{k}$.

Let $\mathbf{t} = (t_1, t_2, \ldots, t_a)$ be a collection of variables. A rational function $\phi \in \mathbb{Q}(\mathbf{t})$ is called $\mathbb{Z}$-*subtraction free* if it can be written as a ratio $P(\mathbf{t})/Q(\mathbf{t})$ where P, Q are polynomials in $\mathbf{t}$ with nonnegative integer coefficients. A rational map $f : \mathbf{k}^a \dashrightarrow \mathbf{k}^b$ is called $\mathbb{Z}$-*subtraction free* if it can be written as $f(\mathbf{x}) = (\phi_1(\mathbf{x}), \phi_2(\mathbf{x}), \ldots, \phi_b(\mathbf{x}))$ where $\phi_1, \ldots, \phi_b$ are $\mathbb{Z}$-subtraction free rational functions. (Such maps are called *admissible* in [Lus19, BH21].)

For $(\mathbf{v}, \mathbf{w}) \in \mathrm{Red}(v, w)$, let $\mathbf{g}_{\mathbf{v}, \mathbf{w}} : (\mathbf{k}^*)^{\ell(w) - \ell(v)} \to \vec{R}_{v,w}^{\circ}(\mathbf{k})$ denote the MR-parametrization. It is known (see [BH21, Proposition 6.1]) that for $(\mathbf{v}, \mathbf{w}), (\mathbf{v}', \mathbf{w}') \in \mathrm{Red}(v, w)$ the rational map $\mathbf{g}_{\mathbf{v}', \mathbf{w}'}^{-1} \circ \mathbf{g}_{\mathbf{v}, \mathbf{w}} : (\mathbf{k}^*)^{\ell(w) - \ell(v)} \dashrightarrow (\mathbf{k}^*)^{\ell(w) - \ell(v)}$ is $\mathbb{Z}$-subtraction free, and this rational function does not depend on $\mathbf{k}$. Similarly, we obtain MR-parametrizations of $\overleftarrow{R}_{v,w}^{\circ}(\mathbf{k})$ by

considering *leftmost* distinguished subexpressions for v inside $\mathbf{w}$, i.e., by applying the map $gB_- \mapsto B_- g^\iota$ of (3.5).

We say that an isomorphism $\phi : \vec{R}^\circ_{v,w}(\mathbf{k}) \xrightarrow{\sim} \vec{R}^\circ_{v,w}(\mathbf{k})$ is $\mathbb{Z}$-*subtraction free* if the composition

$$(10.1) \qquad\qquad \mathbf{g}_{\mathbf{v},\mathbf{w}}^{-1} \circ \phi \circ \mathbf{g}_{\mathbf{v},\mathbf{w}} : (\mathbf{k}^*)^{\ell(w)-\ell(v)} \dashrightarrow (\mathbf{k}^*)^{\ell(w)-\ell(v)}$$

is $\mathbb{Z}$-subtraction free. This notion does not depend on the choice of $(\mathbf{v}, \mathbf{w})$. Similarly, we define the notion of $\mathbb{Z}$-*subtraction free* for isomorphisms $\phi : \vec{R}^\circ_{v,w}(\mathbf{k}) \to \overleftarrow{R}^\circ_{v,w}(\mathbf{k})$, and so on.

Theorem 10.1. *The chiral maps $\vec{\chi}_v, \overleftarrow{\chi}_v$, the opposite chiral maps $\vec{\chi}^w, \overleftarrow{\chi}^w$, the pre-twists $\vec{\tau}^{\mathrm{pre}}_{v,w}, \overleftarrow{\tau}^{\mathrm{pre}}_{v,w}$, the twists $\vec{\tau}_{v,w}, \overleftarrow{\tau}_{v,w}$, and the reversal maps $\mathrm{r\vec{e}v}, \mathrm{r\overleftarrow{e}v}$ are $\mathbb{Z}$-subtraction free. The corresponding $\mathbb{Z}$-subtraction free rational functions in $\mathbb{Q}(\mathbf{t})$ appearing in (10.1) are independent of $\mathbf{k}$.*

Now let $K \subset \mathbf{k}$ be a semifield. For $v \le w$ in W, we let

$$\vec{R}^\circ_{v,w}(K) := \{\mathbf{g}_{\mathbf{v},\mathbf{w}}(\mathbf{t})B_-(\mathbf{k}) \mid \mathbf{t} \in K^{J^\circ_{\mathbf{v}}}\} \quad \subset (G/B_-)(\mathbf{k}).$$

As we mentioned above, the subset $\vec{R}^\circ_{v,w}(K) \subset (G/B_-)(\mathbf{k})$ does not depend on the choice of $(\mathbf{v}, \mathbf{w}) \in \mathrm{Red}(v, w)$ by [BH21, Proposition 6.1]; see also the last sentence of [BH21, Section 3]. Similarly, we define $\overleftarrow{R}^\circ_{v,w}(K) \subset (B_- \backslash G)(\mathbf{k})$ as the image of $\vec{R}^\circ_{v,w}(K)$ under the map $gB_- \mapsto B_- g^\iota$. We let

$$(G/B_-)(K) := \bigsqcup_{v \le w} \vec{R}^\circ_{v,w}(K) \quad \text{and} \quad (B_- \backslash G)(K) := \bigsqcup_{v \le w} \overleftarrow{R}^\circ_{v,w}(K).$$

Theorem 10.1 has the following immediate corollary.

Corollary 10.2. *The chiral maps $\vec{\chi}_v, \overleftarrow{\chi}_v$, the opposite chiral maps $\vec{\chi}^w, \overleftarrow{\chi}^w$, the pre-twists $\vec{\tau}^{\mathrm{pre}}_{v,w}, \overleftarrow{\tau}^{\mathrm{pre}}_{v,w}$, the twists $\vec{\tau}_{v,w}, \overleftarrow{\tau}_{v,w}$, and the reversal maps $\mathrm{r\vec{e}v}, \mathrm{r\overleftarrow{e}v}$ are bijections between the respective semifield points.*

Remark 10.3. Using the $\mathbb{Z}$-subtraction free rational maps in Theorem 10.1, we may extend the definitions of these maps to flag manifolds and Richardson varieties defined over arbitrary semifields; see [BH21, Section 4].

Our proof of Theorem 10.1 will rely on the results of [BH21]. Let $\dot{G}$ be a simply-connected, semisimple, complex algebraic group with simply-laced root datum that folds to G. Thus $\dot{G}$ is equipped with an automorphism $\sigma : \dot{G} \to \dot{G}$ so that $\dot{G}^\sigma \cong G$. We fix a dominant regular integral weight λ for $\dot{G}$. Let $^\lambda\dot{V}$ denote the integrable highest weight module of the quantum group with simply laced root datum folding to G, defined over $\mathcal{A} := \mathbb{Z}[q, q^{-1}]$. The $\mathcal{A}$-module $^\lambda\dot{V}$ is free over $\mathcal{A}$ with canonical basis $\dot{\mathbf{B}}(\lambda)$. Let

$$\dot{V}(\mathbf{k}) := {}^\lambda\dot{V} \otimes_{\mathcal{A}} \mathbf{k},$$

where the map $\mathcal{A} \to \mathbf{k}$ is given by $q \mapsto 1$. The $\mathbf{k}$-vector space $\dot{V}(\mathbf{k})$ is equipped with the image of the canonical basis, which we also denote by $\dot{\mathbf{B}}(\lambda)$. We consider the projectivization $\mathbb{P}(\dot{V}(\mathbf{k}))$ of $\dot{V}(\mathbf{k})$. For $\xi = \sum_{b \in \dot{\mathbf{B}}(\lambda)} \xi_b b$, we denote by $[\xi]$ the corresponding element of $\mathbb{P}(\dot{V}(\mathbf{k}))$, for which we will be interested in the ratios $\xi_b/\xi_{b'}$. See [BH21, Section 2G] for background and further details.

The flag variety $G/B_-(\mathbf{k})$ can be identified with a subvariety of $\mathbb{P}(\dot{V}(\mathbf{k}))$; see [BH21, Equation (2-2)]. Thus, $\vec{R}^\circ_{v,w}(\mathbf{k})$ can be identified with a locally closed subvariety

$$\vec{P}_{v,w}(\mathbf{k})^\sigma \subset \mathbb{P}(\dot{V}(\mathbf{k})).$$

By [BH21, Theorem 3.8], $\vec{R}^\circ_{v,w}(K)$ is identified with the subset

$$\{[\xi] \in \vec{P}_{v,w}(\mathbf{k})^\sigma \mid \xi = \sum_{b \in \dot{\mathbf{B}}(\lambda)} \xi_b b, \text{ where } \xi_b/\xi_{b'} \in K \text{ for all } b, b' \in \dot{\mathbf{B}}(\lambda) \text{ such that } \xi_b, \xi_{b'} \neq 0\}.$$

We will similarly identify $\overleftarrow{R}^\circ_{v,w}(\mathbf{k}) \supset \overleftarrow{R}^\circ_{v,w}(K)$ with $\overleftarrow{P}_{v,w}(\mathbf{k})^\sigma \supset \overleftarrow{P}_{v,w}(K)^\sigma$ inside $\mathbb{P}(\dot{V}(\mathbf{k}))$.

Proof of Theorem 10.1. Let $\phi : \vec{R}^\circ_{v,w}(\mathbf{k}) \xrightarrow{\sim} \vec{R}^\circ_{v,w}(\mathbf{k})$ be an isomorphism, and suppose that $[\xi'] = \phi([\xi])$. Then the ratios $\xi'_{b'_1}/\xi'_{b'_2}$ can be expressed as rational functions of ξ_{b_1}/ξ_{b_2}, where $b_1, b_2, b'_1, b'_2 \in \dot{\mathbf{B}}(\lambda)$. For any of the maps we consider, these rational functions are represented by ratios of integer polynomials independent of $\mathbf{k}$. It follows that the rational maps considered in Theorem 10.1 are independent of $\mathbf{k}$.

Chiral map. Our proof of Theorem 2.3 gives an isomorphism

$$\overleftarrow{\chi}_v : \vec{R}^\circ_{v,w_0}(K) \xrightarrow{\sim} \overleftarrow{R}^\circ_{v,w_0}(K).$$

(The semifield analog of Lemma 6.1 is [BH21, Theorem 1.1(3)].) However, the remainder of the proof used continuity arguments. We instead proceed as follows. Choose a reduced expression $\mathbf{i} = (i_1, \ldots, i_l)$ for $w_0 w^{-1}$ and for $\mathbf{t} = (t_1, \ldots, t_l)$, let $\mathbf{x_i}(\mathbf{t}) := x_{i_1}(t_1) \cdots x_{i_l}(t_l)$. Here $l = \ell(w_0) - \ell(w)$. We have an isomorphism

$$K^{\ell(w_0)-\ell(w)} \times \vec{R}^\circ_{v,w}(K) \xrightarrow{\sim} \vec{R}^\circ_{v,w_0}(K), \quad (\mathbf{t}, gB_-) \mapsto \mathbf{x_i}(\mathbf{t})gB_-.$$

Pick an MR-parametrization $K^{\ell(w)-\ell(v)} \xrightarrow{\sim} \vec{R}^\circ_{v,w}(K)$ sending $\mathbf{t}' \mapsto \mathbf{g}_{\mathbf{v},\mathbf{w}}(\mathbf{t}')$. We thus obtain a parametrization

$$f : K^{\ell(w_0)-\ell(w)} \times K^{\ell(w)-\ell(v)} \xrightarrow{\sim} \vec{R}^\circ_{v,w_0}(K), \quad (\mathbf{t}, \mathbf{t}') \mapsto \mathbf{x_i}(\mathbf{t})\mathbf{g}_{\mathbf{v},\mathbf{w}}(\mathbf{t}')B_-.$$

Now, $\overleftarrow{\chi}_v(f(\mathbf{t}, \mathbf{t}')) \in \overleftarrow{R}^\circ_{v,w_0}(K)$, so all ratios $\xi_b/\xi_{b'}$ (with $\xi_b, \xi_{b'} \neq 0$) of the coordinates of $[\xi] = \overleftarrow{\chi}_v(f(\mathbf{t}, \mathbf{t}'))$ belong to K and must be given by $\mathbb{Z}$-subtraction free rational functions in $\mathbb{Q}(\mathbf{t}, \mathbf{t}')$. We now argue that these functions are $\mathbb{Z}$-subtraction free at $\mathbf{t} = 0$.

Let $K^! := K \sqcup \{0\}$. Observe that for $(\mathbf{t}, \mathbf{t}') \in (K^!)^{\ell(w_0)-\ell(w)} \times K^{\ell(w)-\ell(v)}$, we have $f(\mathbf{t}, \mathbf{t}') \in ((BvB_-)/B_-)(\mathbf{k})$, and thus $\overleftarrow{\chi}_v(f(\mathbf{t}, \mathbf{t}')) \in (B_-\backslash(B_-vB))(\mathbf{k})$ is well defined. In particular, after scaling all coordinates, we may assume that for all $b \in \dot{\mathbf{B}}(\lambda)$, we have $\xi_b \in \mathbb{Z}_{\geq 0}[\mathbf{t}, \mathbf{t}']$, and that moreover, not all ξ_b are divisible by t_1. Thus $\overleftarrow{\chi}_v(f(0, t_2, \ldots, t_l, \mathbf{t}'))$ is represented by $[\xi']$ where $\xi'_b = \xi_b|_{t_1=0} \in \mathbb{Z}_{\geq 0}[t_2, \ldots, t_l, \mathbf{t}']$. Repeating this argument for $t_2, t_3, \ldots, t_l$, we deduce that

$$\overleftarrow{\chi}_v(\mathbf{g}_{\mathbf{v},\mathbf{w}}(\mathbf{t}')) = [\xi''], \quad \text{where} \quad \xi''_b \in \mathbb{Z}_{\geq 0}[\mathbf{t}'] \quad \text{for all } b \in \dot{\mathbf{B}}(\lambda).$$

We deduce that $\overleftarrow{\chi}_v$ restricts to a bijection $\overleftarrow{\chi}_v : \vec{R}^\circ_{v,w}(K) \xrightarrow{\sim} \overleftarrow{R}^\circ_{v,w}(K)$ for all $v \leq w$. By the Marsh–Rietsch Chamber Ansatz (cf. Theorem 5.1 and [BH21, Remark 2.12]), we deduce that the map $\overleftarrow{\chi}_v : \vec{R}^\circ_{v,w}(K) \xrightarrow{\sim} \overleftarrow{R}^\circ_{v,w}(K)$ is $\mathbb{Z}$-subtraction free. The proof for the map $\vec{\chi}_v$ is similar.

Reversal maps. For the map $\text{r\v{e}v}$ sending $gB_- \mapsto g^\theta \dot{w}_0 B_-$, this is [BH21, Theorem 6.4]. For the map $\text{r\v{e}v}$, we note that $B_- g \mapsto g^\iota B_-$ induces, by definition, an isomorphism $(B_- \backslash G)(K) \to (G/B_-)(K)$. The claim then follows in the same way as in the proof of Proposition 6.4.

Opposite chiral map. Replacing $\vec{R}^{>0}_{v,w}$ by $\vec{R}^\circ_{v,w}(K)$, etc. in the proof of Theorem 6.6, we deduce that

$$\vec{\chi}^w : \overleftarrow{\vec{R}}^\circ_{v,w}(K) \xrightarrow{\sim} \vec{R}^\circ_{v,w}(K)$$

is a bijection. The $\mathbb{Z}$-subtraction freeness follows as before.

Pre-twist. The computation in the proof of Theorem 2.6 shows that $\vec{\tau}^{\text{pre}}_{v,w}$ restricts to a bijection $\overleftarrow{R}^\circ_{v,w}(K) \xrightarrow{\sim} \vec{R}^\circ_{v,w}(K)$. The claim follows as before.

Twist. Since the twist map is the composition of the pre-twist and the chiral map, this follows immediately. $\qquad\square$

11. Cluster structures on Richardson varieties

The goal of this section is to use our results to relate the two conjectural cluster structures on $\vec{R}^\circ_{v,w}$ described in [Lec16, Ing19].

Throughout this section, we assume that G is of arbitrary type, even though the results of [Lec16] apply only when G is of simply laced (ADE) type while the results of [Ing19] apply only when G is of type A. We also fix $v \le w \in W$ and $(\mathbf{v}, \mathbf{w}) \in \text{Red}(v, w)$ with $\mathbf{w} = (i_1, i_2, \dots, i_m)$.

11.1. **Cluster algebras.** We refer the reader to [FZ02, FWZ16] for a general introduction to cluster algebras. A *quiver* is a directed graph Q without directed cycles of size 1 and 2. An *ice quiver* is a quiver Q together with a partition of its vertex set into two sets $V(Q) = V_f(Q) \sqcup V_m(Q)$ of *frozen* and *mutable* vertices, respectively. A *seed* is a pair $(Q, \mathbf{x})$ where $\mathbf{x} = (x_j)_{j \in V(Q)}$ is a collection of algebraically independent elements of the *ambient field* $\mathcal{F} = \mathbb{C}(y_1, \dots, y_r)$ of rational functions in some number of variables. For a mutable vertex $v \in V_m(Q)$, the *mutation* μ_v is an operation that changes a seed $(Q, \mathbf{x})$ into another seed $(Q', \mathbf{x}')$. The elements of all seeds that can be obtained from $(Q, \mathbf{x})$ via a sequence of mutations are called *cluster variables*, where $(x_j)_{j \in V_f(Q)}$ are referred to as *frozen variables*.

The *cluster algebra* $\mathcal{A}(Q, \mathbf{x})$ is the subring of $\mathcal{F}$ generated by all cluster variables, frozen variables, and inverses of frozen variables. A *cluster algebra structure* on a variety X is a ring isomorphism $\phi : \mathcal{A}(Q, \mathbf{x}) \xrightarrow{\sim} \mathbb{C}[X]$ from some cluster algebra to the ring of regular functions on X. Under this isomorphism, each cluster variable becomes a regular function on X. Clearly, the isomorphism ϕ is determined by its values on the elements of the initial seed $\mathbf{x}$.

As we mentioned in the introduction, cluster structures have been previously constructed on special families of open Richardson varieties, namely, on unipotent cells, double Bruhat cells in type A, and open positroid varieties; see [BFZ05, Sco06, GLS13, SSBW19, GL19, GY20].

11.2. **Leclerc's construction.** For each $r \in [m]$, recall that we have a regular function $\vec{f}_r : N \to \mathbb{C}$ given by (2.9). It is not in general irreducible as an element of the coordinate ring $\mathbb{C}[N]$. When G is of simply laced type, by [Lec16, Corollary 4.4], there are exactly $\ell(w) - \ell(v)$ irreducible factors of $\prod_{r \in [m]} \vec{f}_r$, and they form the initial seed in Leclerc's conjectural cluster algebra structure on $\vec{R}_{v,w}^\circ$. The following conjecture applies to G of arbitrary type, but is open even when G is simply laced.

Conjecture 11.1. *There exists a unique family $(F_r)_{r \in J_{\mathbf{v}}^\circ}$ of irreducible elements of $\mathbb{C}[N]$ such that for each $r \in [m]$, we have*

$$\vec{f}_r = \prod_{j \in J_{\mathbf{v}}^\circ : j \geq r} F_j^{p_{r,j}}$$

for some nonnegative integers $p_{r,j} \geq 0$ satisfying $p_{r,r} = 1$ for all $r \in J_{\mathbf{v}}^\circ$.

So far, the functions $\vec{f}_r$ are defined on N. We explain how to convert them into functions on the flag variety. Leclerc works with the left-sided quotient $B_- \backslash G$. In order to view each $\vec{f}_r$ as a function on $\overleftarrow{R}_{v,w}^\circ$, he uses the isomorphism (2.2) between $\overleftarrow{R}_{v,w}^\circ$ and $(N\dot{v} \cap \dot{v}N)^{\cap w}$. Specifically, each $\vec{f}_r$ is a function on $\overleftarrow{R}_{v,w}^\circ$ sending $B_-\dot{v}h \mapsto \vec{f}_r(h)$ for $h \in \dot{v}^{-1}(N\dot{v}\cap\dot{v}N)^{\cap w} \subset N$. By Lemma 3.3, we have $\vec{f}_r(yh) = \vec{f}_r(h)$ for all $y \in N \cap \dot{v}^{-1}N_-\dot{v}$. By Lemma 3.2, the multiplication map gives an isomorphism

$$(N \cap \dot{v}^{-1}N_-\dot{v}) \times \dot{v}^{-1}(N\dot{v} \cap \dot{v}N)^{\cap w} \xrightarrow{\sim} \dot{v}^{-1}(\dot{v}N)^{\cap w}.$$

(Here we have used the fact that left multiplication by $N \cap \dot{v}^{-1}N_-\dot{v}$ preserves the subset $\dot{v}^{-1}B_-wB_-$.) Thus $\vec{f}_r$ sends $B_-\dot{v}h \mapsto \vec{f}_r(h)$ where h is now any element of $\dot{v}^{-1}(\dot{v}N)^{\cap w}$. Recall from Proposition 2.4 that the element $\vec{y} := [g^T\dot{w}]_L^+$ satisfies $\vec{y} \in \dot{v}^{-1}(\dot{v}N)^{\cap w}$. We see that the right pre-twist isomorphism from Definition 2.5 is exactly the missing ingredient needed to define Leclerc's functions on the right-sided quotient G/B_-.

Definition 11.2. We consider Leclerc's minors $\vec{f}_r$ as regular functions on $\vec{R}_{v,w}^\circ$, sending $gB_- \mapsto \vec{f}_r(\vec{y})$, where $g \in N\dot{v}$ and $\vec{y} := [g^T\dot{w}]_L^+$.

When G is of simply laced type, Leclerc [Lec16] defined a quiver Q_{Lec} on $\ell(w) - \ell(v)$ vertices in terms of irreducible morphisms of preprojective algebra modules. He showed that the resulting seed gives rise to a cluster subalgebra of $\mathbb{C}[\vec{R}_{v,w}^\circ]$ and conjectured that this subalgebra coincides with $\mathbb{C}[\vec{R}_{v,w}^\circ]$. Assuming Conjecture 11.1, the vertex set of Q_{Lec} is naturally identified with $J_{\mathbf{v}}^\circ$.

11.3. **Ingermanson's construction.** We continue to assume that G is of arbitrary type. The results of [Ing19] apply when $G = \text{SL}_n(\mathbb{C})$. However, the constructions below can be extended to the case of arbitrary G. We note that [Ing19] works inside G/B rather than G/B_-, and thus make the necessary modifications (such as applying the involution $x \mapsto x^\theta$ from Section 3.2) to match her notation to our notation.

Following [Ing19, Proposition V.3], let us consider a labeling of the chambers of the wiring diagram for $\mathbf{w}$ by minors which are very closely related to the functions $\overleftarrow{f}_r$ defined in (2.12). Specifically, set

$$(11.1) \qquad\qquad \overleftarrow{f}_r' := \Delta_{v_{(r-1)}w_0\omega_{i_r^*}}^{w_{(r-1)}w_0\omega_{i_r^*}} : B \to \mathbb{C}.$$
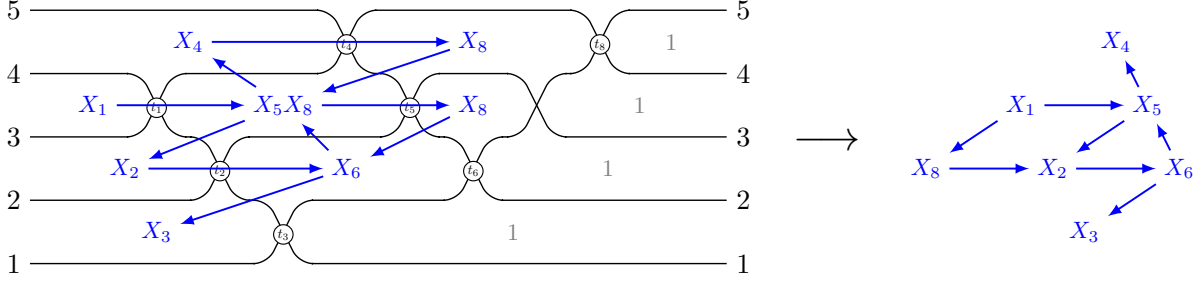
FIGURE 3. Constructing the quiver of [Ing19] from the BFZ quiver by combining the arrows between the irreducible factors; cf. [Ing19, Figure 7.8].

When $G = \mathrm{SL}_n(\mathbb{C})$, we therefore have

$$(11.2) \qquad \vec{f}_r = \vec{\Delta}_{C_r^{\blacktriangleright}}, \quad \overleftarrow{f}_r = \overleftarrow{\Delta}_{C_r^{\blacktriangleleft}}, \quad \text{and} \quad \overleftarrow{f}'_r = \overleftarrow{\Delta}_{C_r^{\blacktriangleright}};$$

cf. (2.15), Section 2.9, and Example 11.7.

Following [Ing19, Notation III.28], we choose a representative for gB_- which differs from the z-matrix in Theorem 5.1 by a torus element; cf. (11.4).

Definition 11.3. Let $gB_- \in \vec{R}^{\circ}_{v,w}$. Let $z_{\mathrm{Ing}} \in B_-$ be the unique matrix satisfying $z_{\mathrm{Ing}} w B_- = gB_-$ and $\Delta^{ww_0\omega_{i^*}}_{vw_0\omega_{i^*}}(z^T_{\mathrm{Ing}}) = 1$ for all $i \in I$. Each $\overleftarrow{f}'_r$ is considered as a regular function on $\vec{R}^{\circ}_{v,w}$ sending $gB_- \mapsto \overleftarrow{f}'_r(z_{\mathrm{Ing}})$.

Remark 11.4. In terms of the usual matrix minors when $G = \mathrm{SL}_n(\mathbb{C})$, we have $\Delta^{ww_0\omega_{i^*}}_{vw_0\omega_{i^*}} = \Delta^{w[i+1,n]}_{v[i+1,n]}$ and $\overleftarrow{f}'_r = \Delta^{w_{(r-1)}[i_r+1,n]}_{v_{(r-1)}[i_r+1,n]}$.

The cluster variables $(X_j)_{j \in J^{\circ}_{\mathbf{v}}}$ introduced in [Ing19, Proposition V.1] are irreducible factors of these regular functions. There are exactly $\ell(w) - \ell(v)$ of them, and [Ing19, Section 4] describes a combinatorial procedure for how each function $\overleftarrow{f}'_r$ factorizes into irreducibles. In particular, these functions are labeled by the elements of $J^{\circ}_{\mathbf{v}}$ and Conjecture 11.1 becomes manifestly true in this description.

[Ing19, Definition VII.2] gives an entirely different construction of the quiver Q_{Ing}: it is simply obtained from the *BFZ quiver* [BFZ05, Definition 2.2] associated to the wiring diagram of $\mathbf{w}$ by summing up the contributions of all arrows between each pair of irreducible factors and removing all loops and 2-cycles; see Figure 3 and Example 11.7. The following conjecture is supported by vast computational evidence.

Conjecture 11.5. *Let $G = \mathrm{SL}_n(\mathbb{C})$.*

  (1) *The quivers Q_{Lec} and Q_{Ing} are isomorphic.*
  (2) *Assuming Conjecture 11.1, the isomorphism is given by naturally identifying both vertex sets with $J^{\circ}_{\mathbf{v}}$.*
  (3) *For each $v \le w \in W$, these quivers are locally acyclic in the sense of [Mul13].*

We shall return to the study of the cluster structure of Richardson and related varieties in joint work with Melissa Sherman-Bennett and David Speyer.[2]

11.4. **Relating the two cluster structures.**

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[2]We understand that Khrystyna Serhiyenko and Melissa Sherman-Bennett have a forthcoming work that verifies some parts of the Conjecture 11.5.
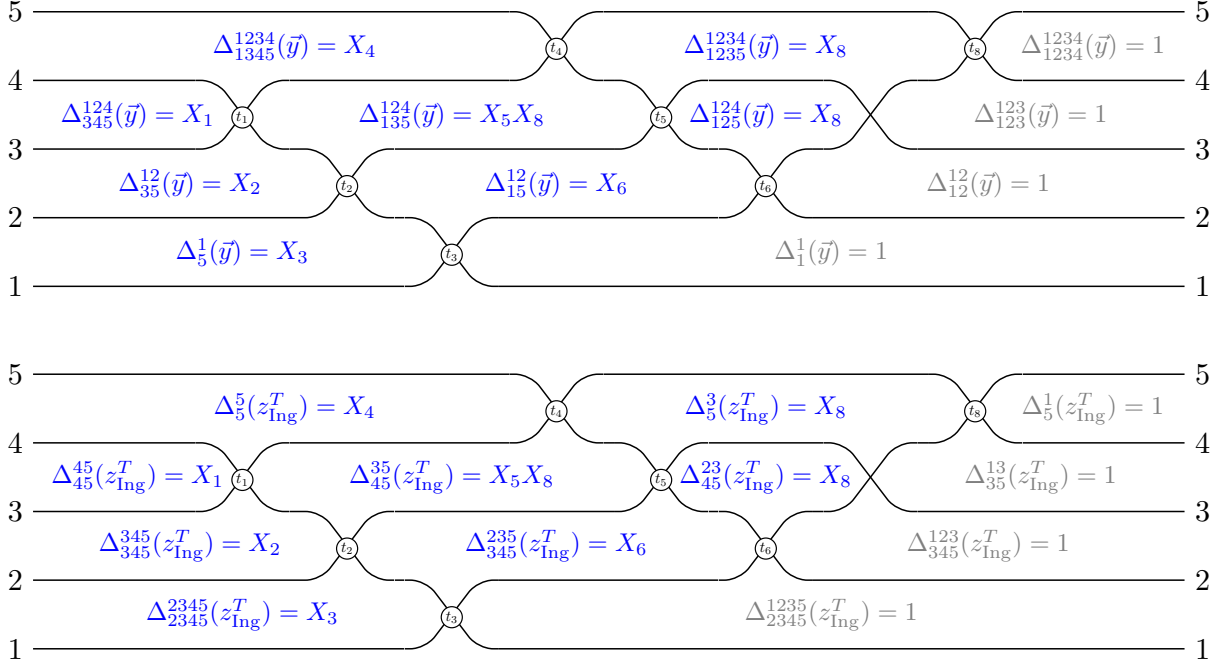
FIGURE 4. Comparing the conjectural cluster structures of [Lec16] and of [Ing19]; cf. Theorem 11.6 and Example 11.7.

Theorem 11.6. *The functions $\vec{f}_r$ and $\overleftarrow{f}'_r$ coincide as regular functions on $\vec{R}^\circ_{v,w}$. More precisely, in the notation of Definitions 11.2 and 11.3, we have*

$$(11.3) \qquad \overleftarrow{f}'_r(z^T_{\mathrm{Ing}}) = \vec{f}_r(\vec{y}) \quad \text{for each } r \in [m].$$

Example 11.7. Let us consider the running example of [Ing19]. We have $n = 5$, $v = s_3$, $\mathbf{w} = (3, 2, 1, 4, 3, 2, 3, 4)$, $J^\circ_{\mathbf{v}} = \{1, 2, 3, 4, 5, 6, 8\}$. The MR-parameters $(t_j)_{j \in J^\circ_{\mathbf{v}}}$ are expressed in terms of the cluster variables $(X_j)_{j \in J^\circ_{\mathbf{v}}}$ by applying the Chamber Ansatz (2.16) to the chamber labels in Figure 3:

$$t_1 = \frac{X_2 X_4}{X_1 X_5 X_8}, \quad t_2 = \frac{X_3 X_5 X_8}{X_2 X_6}, \quad t_3 = \frac{X_6}{X_3}, \quad t_4 = \frac{X_5}{X_4}, \quad t_5 = \frac{X_6}{X_5 X_8}, \quad t_6 = \frac{X_8}{X_6}, \quad t_8 = \frac{1}{X_8}.$$

With this substitution, we compute

$$\vec{y} = \begin{pmatrix} 1 & \frac{X_3 X_4 X_5 + X_2 X_4 + X_1 X_6}{X_4 X_5 X_6} & \frac{X_3 X_5 + X_2}{X_6} & \frac{(X_1 + X_2) X_3}{X_6} & X_3 \\ 0 & 1 & X_5 & \frac{(X_1 + X_2) X_6}{X_2 X_8} & X_6 \\ 0 & 0 & 1 & \frac{X_2 X_4 + X_1 X_6}{X_2 X_5 X_8} & 0 \\ 0 & 0 & 0 & 1 & X_8 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad z^T_{\mathrm{Ing}} = \begin{pmatrix} \frac{1}{X_3} & \frac{X_3 X_5 + X_2}{X_2 X_6} & \frac{X_1 + X_2}{X_1 X_8} & \frac{X_5}{X_4} & 1 \\ 0 & \frac{X_3}{X_2} & 0 & -\frac{X_1 X_6}{X_2 X_4} & -\frac{X_2 X_4 + X_1 X_6}{X_2 X_5} \\ 0 & 0 & \frac{X_2}{X_1} & \frac{X_5 X_8}{X_4} & X_8 \\ 0 & 0 & 0 & \frac{X_1}{X_4} & 0 \\ 0 & 0 & 0 & 0 & X_4 \end{pmatrix}.$$

One can check that the matrix z^θ_{Ing} indeed coincides with the matrix given in [Ing19, Example VII.27]. The minors of $\vec{y}$ and z^T_{Ing} appearing in Theorem 11.6 are compared in Figure 4.

Proof of Theorem 11.6. Let $(y_+, y_0, y_-) \in N \times H \times N_-$ be given by (7.1). By (7.2), we have $y_+ = \vec{y}$. Let $z := (\dot{w} y_- \dot{w}^{-1})^T$ be given by (8.6). As explained after (8.6), we have $z \in N_-$ and $z w B_- = g B_-$.

Our first goal is to show that z differs from z_{Ing} by a torus element. Specifically, we claim that

$$(11.4) \qquad z_{\mathrm{Ing}} := (\dot{w}y_0 y_- \dot{w}^{-1})^T = z \cdot \dot{w}y_0\dot{w}^{-1}$$

satisfies the conditions of Definition 11.3. First, we have $z \in N_-$ and $\dot{w}y_0\dot{w}^{-1} \in H$, so $z_{\mathrm{Ing}} \in B_-$. Second, we have $z_{\mathrm{Ing}}wB_- = z\dot{w}y_0 B_- = zwB_- = gB_-$. It remains to show that $\Delta^{ww_0\omega_{i*}}_{vw_0\omega_{i*}}(z_{\mathrm{Ing}}^T) = 1$ for all $i \in I$. By Eq. (3.10), we have

$$\Delta^{ww_0\omega_{i*}}_{vw_0\omega_{i*}}(z_{\mathrm{Ing}}^T) = \Delta^{ww_0\omega_{i*}}_{vw_0\omega_{i*}}(\dot{w}y_0\dot{w}^{-1} \cdot z^T) = \Delta^{ww_0\omega_{i*}}_{ww_0\omega_{i*}}(\dot{w}y_0\dot{w}^{-1}) \cdot \Delta^{ww_0\omega_{i*}}_{vw_0\omega_{i*}}(z^T).$$

We compute each of the two factors separately. Applying (3.11) and then (3.9), we get

$$\Delta^{ww_0\omega_{i*}}_{ww_0\omega_{i*}}(\dot{w}y_0\dot{w}^{-1}) = \Delta^{w\omega_i}_{w\omega_i}((\dot{w}y_0\dot{w}^{-1})^\iota) = \Delta^{w\omega_i}_{w\omega_i}(\overline{w}y_0^{-1}\overline{\overline{w^{-1}}}) = \Delta^{\omega_i}_{\omega_i}(\overline{\overline{w^{-1}}}\,\overline{w}y_0^{-1}\overline{\overline{w^{-1}}}\,\overline{w}) = \Delta^{\omega_i}_{\omega_i}(y_0^{-1}).$$

It suffices to assume that $g = \mathbf{g}_{\mathbf{v},\mathbf{w}}(\mathbf{t})$ is MR-parametrized for $\mathbf{t} \in (\mathbb{R}_{>0})^{J^\circ_{\mathbf{v}}}$, since $\vec{R}^{>0}_{v,w}$ is Zariski dense inside $\vec{R}^\circ_{v,w}$. Then the first factor $\Delta^{ww_0\omega_{i*}}_{ww_0\omega_{i*}}(\dot{w}y_0\dot{w}^{-1})$ is given by the inverse of the right hand side of (7.3). The second factor $\Delta^{ww_0\omega_{i*}}_{vw_0\omega_{i*}}(z^T)$ coincides with $\overleftarrow{f}_j(z^T)$ where j is the maximal index such that $i_j = i$. Applying (5.2), we indeed see that $\Delta^{ww_0\omega_{i*}}_{vw_0\omega_{i*}}(z_{\mathrm{Ing}}^T) = 1$. We have shown (11.4).

The proof of (11.3) proceeds by an entirely similar argument. We have

$$\overleftarrow{f}'_r(z_{\mathrm{Ing}}^T) = \Delta^{w_{(r-1)}w_0\omega_{i_r^*}}_{v_{(r-1)}w_0\omega_{i_r^*}}(\dot{w}y_0\dot{w}^{-1} \cdot z^T) = \Delta^{w_{(r-1)}w_0\omega_{i_r^*}}_{w_{(r-1)}w_0\omega_{i_r^*}}(\dot{w}y_0\dot{w}^{-1}) \cdot \Delta^{w_{(r-1)}w_0\omega_{i_r^*}}_{v_{(r-1)}w_0\omega_{i_r^*}}(z^T).$$

Restricting to the case $gB_- \in \vec{R}^{>0}_{v,w}$ with $g = \mathbf{g}_{\mathbf{v},\mathbf{w}}(\mathbf{t})$ as above, the first factor is computed using Lemma 7.1 and the second factor is computed using (5.2). Computing the right-hand side of (11.3) using (2.11), we deduce the result. $\qquad \square$

APPENDIX A. FIXED POINTS OF THE TWIST

The twist map $\vec{\tau}_{v,w}$ is an automorphism of $\vec{R}^\circ_{v,w}$ which preserves the totally positive part $\vec{R}^{>0}_{v,w}$ and generalizes the right twist of [MuSp17]. In [Kar19, Section 4.5], Karp posed the problem of finding the fixed points of the right twist map on open positroid varieties and their totally positive parts. This section is motivated by a related question for open Richardson varieties.

Question A.1. Does $\vec{\tau}_{v,w} : \vec{R}^{>0}_{v,w} \xrightarrow{\sim} \vec{R}^{>0}_{v,w}$ have a unique fixed point for all $v \leq w \in W$?

We show below that in general, the answer is "no" (even for open positroid varieties). We start with an interesting example where the answer appears to be "yes."

Conjecture A.2. Let $G = \mathrm{SL}_n(\mathbb{C})$, $v = \mathrm{id}$, and $w = w_0$. Then the unique fixed point of $\vec{\tau}_{v,w}$ inside $\vec{R}^{>0}_{v,w}$ equals gB_- where $g = (g_{i,j})^n_{i,j=1}$ is an upper triangular matrix given by

$$g_{i,j} = \sqrt{\binom{n-i}{j-i}\binom{j-1}{j-i}}, \quad \text{for } 1 \leq i \leq j \leq n.$$

Example A.3. For $n = 5$, the above matrix g is given by

$$g = \begin{pmatrix} \sqrt{1} & \sqrt{4} & \sqrt{6} & \sqrt{4} & \sqrt{1} \\ 0 & \sqrt{1} & \sqrt{6} & \sqrt{9} & \sqrt{4} \\ 0 & 0 & \sqrt{1} & \sqrt{6} & \sqrt{6} \\ 0 & 0 & 0 & \sqrt{1} & \sqrt{4} \\ 0 & 0 & 0 & 0 & \sqrt{1} \end{pmatrix}.$$

We now give an example where the twist $\vec{\tau}_{v,w}$ has no fixed points on $\vec{R}_{v,w}^{>0}$.

Example A.4. Consider the open positroid variety $\Pi_{v,w}^{\circ}$ from [MuSp17, Section A.2]. Its Zariski closure is the subvariety of $\mathrm{Gr}(4, 8)$ given by the vanishing of four Plücker coordinates

$$\Delta_{1234} = \Delta_{3456} = \Delta_{5678} = \Delta_{1278} = 0.$$

The frozen variables are given by

$$\Delta_{1238}, \Delta_{2348}, \Delta_{2345}, \Delta_{2456}, \Delta_{4567}, \Delta_{4678}, \Delta_{1678}, \Delta_{1268}.$$

The right twist inverts the frozen variables by [MuSp17, Equation (9)]. Thus, in order for them to stay preserved (and be positive real), all frozen variables have to be equal to 1. There are also five mutable cluster variables

$$\Delta_{1248}, \Delta_{2346}, \Delta_{4568}, \Delta_{2678}, \Delta_{2468}.$$

Using [MuSp17, Equations (A.1)–(A.2)], one can write down a system of equations on these five variables in order for them to stay preserved under the twist map, and check that it admits no positive real solutions.

Remark A.5. The above example is also an example of the twist map having infinite order on the torus quotient $\Pi_{v,w}^{\circ}/H$. This suggests an interesting interplay between the behavior of totally positive fixed points of the twist map and its periodicity properties; see [GP19] for related results.

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