TRIANGULOIDS AND TRIANGULATIONS OF ROOT POLYTOPES

PAVEL GALASHIN, GLEB NENASHEV, AND ALEXANDER POSTNIKOV

Abstract. Triangulations of a product of two simplices and, more generally, of root polytopes are closely related to Gelfand-Kapranov-Zelevinsky's theory of discriminants, to tropical geometry, tropical oriented matroids, and to generalized permutohedra. We introduce a new approach to these objects, identifying a triangulation of a root polytope with a certain bijection between lattice points of two generalized permutohedra. In order to study such bijections, we define *trianguloids* as edge-colored graphs satisfying simple local axioms. We prove that trianguloids are in bijection with triangulations of root polytopes.

1. Introduction

Triangulations of a product $\Delta^{m-1} \times \Delta^{n-1}$ of two simplices have been studied for the last several decades, see e.g. [ES52, Section 8], [FF16, Section 16.3], or [BCS88]. Since then, these objects have naturally appeared in many diverse contexts in combinatorics and algebraic geometry [SZ93, BZ93, GKZ08, BB98, San00]. They have recently become a subject of active research due to their close relationship to tropical geometry [DS04, AD09] and Schubert calculus [AB07].

Triangulations of $\Delta^{m-1} \times \Delta^{n-1}$ are in bijection with various objects, such as *fine mixed subdivisions* of $n\Delta^{m-1}$ [San05, HRS00], *tropical oriented matroids* [AD09, OY11], *tropical pseudohyperplane arrangements* [Hor16], *matching ensembles* [BZ93, OY15], and *compatible families of trees* [Pos09]. In particular, it was shown in [Pos09] that a triangulation of $\Delta^{m-1} \times \Delta^{n-1}$ gives rise to a bijection between lattice points of $(n-1)\Delta^{m-1}$ and of $(m-1)\Delta^{n-1}$. More generally, for an arbitrary connected subgraph G of the complete bipartite graph $K_{m,n}$, [Pos09] introduced the *root polytope* Q_G which specializes to $\Delta^{m-1} \times \Delta^{n-1}$ for $G = K_{m,n}$. He showed that a triangulation τ of Q_G corresponds to a fine mixed subdivision of a *generalized permutohedron* P_G and yields a bijection ϕ_τ between the lattice points of two *trimmed generalized permutohedra* P_G^- and $P_{G^*}^-$. One of the main motivations for this project was to study the bijections ϕ_τ that arise in this way. For example, it follows as a simple consequence of our approach that a triangulation τ of Q_G can be uniquely reconstructed from the corresponding bijection ϕ_τ. See [BZ93, Theorem 5] and [SZ93, Conjecture 6.11] for related results.

Another motivation comes from the work of Ardila and Billey [AB07] who described the matroid formed by the lines in the intersection lattice of m generic complete flags in

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$\mathbb{C}^n$. They raised the *Spread Out Simplices Conjecture* which characterizes the positions of special simplices in a mixed subdivision of $n\Delta^{m-1}$.

We introduce certain edge-colored directed graphs called *trianguloids* (an example shown in Figure 1). We define them axiomatically and show that they are in a natural bijective correspondence with triangulations of $\Delta^{m-1} \times \Delta^{n-1}$ (more generally, of Q_G). One aspect in which trianguloids differ from some of the objects listed above is that our axioms are *local*, and in addition, we make no assumptions on the compatibility of the trees appearing in a triangulation. We hope that these properties of our axioms may produce a way of resolving the Spread Out Simplices Conjecture.

Outline. We introduce root polytopes and their triangulations in Section 2, and then we state our main results for the case $Q_G = \Delta^{m-1} \times \Delta^{n-1}$ in Section 3. We explain the relationship between trianguloids and various objects that have been studied before in Section 4. We then formulate our main result for the case of arbitrary G (Theorem 5.6) in Section 5.

For the remaining part of the paper, we concentrate on the proofs. In Section 6, we show that each triangulation gives rise to a trianguloid. In Section 7, we show that each trianguloid gives rise to a triangulation. Finally, in Section 8 we use our machinery to give simple proofs to Theorems 3.7 and 5.7 that a triangulation τ can be reconstructed from ϕ_τ.

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2. Preliminaries

Let us fix integers $m, n \geq 1$ and consider the sets $[m] := \{1, 2, \dots, m\}$ and $[\bar{n}] := \{\bar{1}, \bar{2}, \dots, \bar{n}\}$. Define the *complete bipartite graph* $K_{m,n}$ to be a simple graph with vertex set $V := [m] \cup [\bar{n}]$ and edge set $\{(i, \bar{j}) \mid i \in [m], \bar{j} \in [\bar{n}]\}$. We identify subgraphs of $K_{m,n}$ with their sets of edges. Clearly, a graph $G \subset K_{m,n}$ is determined by the sets $N_{\bar{1}}(G), N_{\bar{2}}(G), \dots, N_{\bar{n}}(G) \subset [m]$, where $N_{\bar{j}}(G) := \{i \in [m] \mid (i, \bar{j}) \in G\}$ is the *neighborhood* of $\bar{j} \in [\bar{n}]$ in G. Throughout, we fix a connected $G \subset K_{m,n}$ and pay special attention to the case $G = K_{m,n}$.

Consider an $(m + n)$-dimensional real vector space with basis

$$e_1, e_2, \dots, e_m, e_{\bar{1}}, e_{\bar{2}}, \dots, e_{\bar{n}} \in \mathbb{R}^{m+n}.$$

For a set $I \subset [m]$, we define $\Delta_I \subset \mathbb{R}^m$ to be the convex hull of the points $\{e_i \mid i \in I\}$. Thus Δ_I is an $(|I| - 1)$-dimensional simplex. We denote by $\Delta^{m-1} := \Delta_{[m]}$ the standard $(m - 1)$-dimensional simplex.

The *root polytope* $Q_G \subset \mathbb{R}^{m+n}$ was introduced in [Pos09] as the convex hull of the points $e_i - e_{\bar{j}}$ for all $(i, \bar{j}) \in G$. When $G = K_{m,n}$ is complete, Q_G is the direct product of two simplices $\Delta_{[m]} \times \Delta_{[\bar{n}]}$, see [Pos09, Section 12].

We now recall the notions of *Minkowski sum* and *Minkowski difference*.

Definition 2.1. For two subsets $A, B \subset \mathbb{R}^k$, define

$$A + B := \{a + b \mid a \in A,\ b \in B\}, \quad A - B := \{c \in \mathbb{R}^k \mid c + B \subset A\}.$$

Note that $A - B$ can be empty, so it is not always the case that $(A - B) + B = A$. However, if A and B are convex polytopes then it is true that $(A + B) - B = A$, see [Pos09, Lemma 11.1]. We define three polytopes $P_G, P_G^-, P_G^\pm \in \mathbb{R}^m_{\geq 0}$ associated with G as

$$P_G := \sum_{\bar{j} \in [\bar{n}]} \Delta_{N_{\bar{j}}(G)}, \quad P_G^- := P_G - \Delta_{[m]}, \quad P_G^\pm := \big(P_G + (-\Delta_{[m]})\big) \cap \mathbb{R}^m_{\geq 0}.$$

Here $-\Delta_{[m]}$ is the convex hull of $\{-e_i \mid i \in [m]\}$.

Thus P_G is a *generalized permutohedron* and P_G^- is a *trimmed generalized permutohedron* in the sense of [Pos09]. The polytope $P_G^\pm$ contains P_G^-. In the case $G = K_{m,n}$, we have $N_{\bar{j}}(G) = [m]$ for each $\bar{j} \in [\bar{n}]$, so $P_G = n\Delta_{[m]}$ and $P_G^- = P_G^\pm = (n-1)\Delta_{[m]}$ are just dilated $(m-1)$-simplices.

Our main focus is the set of *triangulations* of the root polytope Q_G. For a subgraph $F \subset G$, we let Δ_F be the convex hull of $e_i - e_{\bar{j}}$ for all $(i, \bar{j}) \in F$. Then by [Pos09, Lemma 12.5], Δ_F is a simplex in $\mathbb{R}^{m+n}$ if and only if F is a *forest* of G (i.e., a subset of edges of G that contains no cycles). Moreover, the dimension of Δ_F is $|F| - 1$, and thus Δ_F is top-dimensional (that is, $(m + n - 2)$-dimensional) if and only if F is a spanning tree of G.

Definition 2.2. We say that two simplices Δ_F and $\Delta_{F'}$ *intersect by their common face* if $\Delta_F \cap \Delta_{F'} = \Delta_{F \cap F'}$. A *triangulation* τ of Q_G is a simplicial complex such that each simplex is of the form Δ_F for some forest $F \subset G$, any two simplices in τ intersect by their common face, and the union of these simplices is Q_G.

It turns out that the above condition admits a simple combinatorial characterization:

Definition 2.3. We say that two forests F, F' are *compatible* if there does not exist a pair $M \subset F$, $M' \subset F'$ of partial matchings such that $M \neq M'$ but for all $i \in [m]$ and $\bar{j} \in [\bar{n}]$ we have $\deg_i(M) = \deg_i(M')$ and $\deg_{\bar{j}}(M) = \deg_{\bar{j}}(M')$.

Here $\deg_i(M) := |N_i(M)|$ and $\deg_{\bar{j}}(M) := |N_{\bar{j}}(M)|$ denote the *degrees* of i and $\bar{j}$ in M, and a *partial matching* is a subgraph $M \subset G$ such that the degree of every vertex of G in M is at most 1.

Lemma 2.4. *Two simplices Δ_F and $\Delta_{F'}$ intersect by their common face if and only if F and F' are compatible.*

This lemma is a special case of Lemma 6.1.

For a triangulation τ of Q_G, we denote by

$$\mathcal{F}(\tau) := \{F \subset G \mid \Delta_F \in \tau\}$$

the collection of *forests* of τ. Just as any other (pure) simplicial complex, τ is determined by its top-dimensional simplices, so we denote

$$\mathcal{T}(\tau) := \{T \in \mathcal{F}(\tau) \mid T \text{ is a spanning tree of } G\}.$$

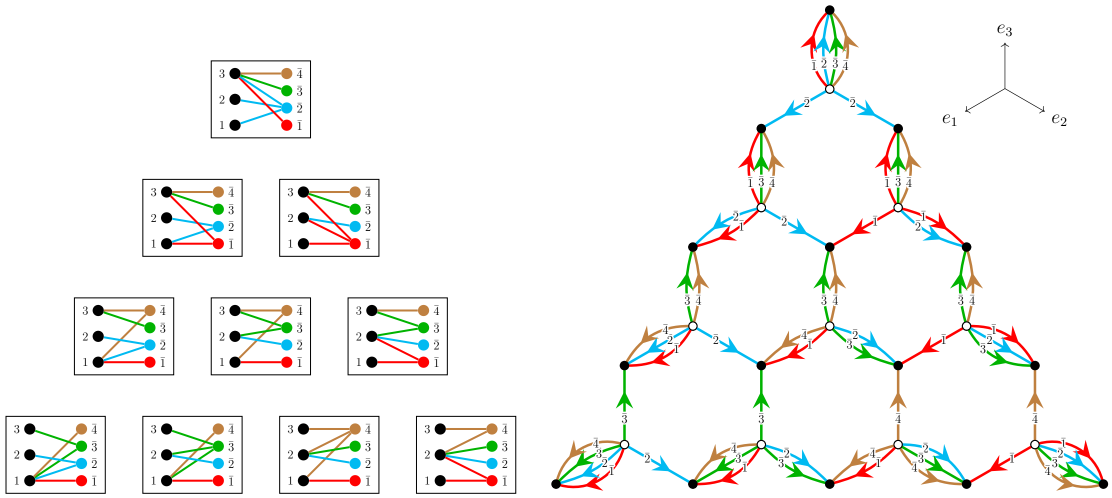

FIGURE 1. A family $\mathcal{T}(\tau)$ of trees for a triangulation τ of $Q_{K_{m,n}}$ for $m = 3, n = 4$ (left), and the corresponding trianguloid $\mathbb{T} := \mathbb{T}_\tau$ (right). The white (resp., black) vertices of $\mathbb{T}$ are the lattice points of $P_G^- = (n-1)\Delta_{[m]}$ (resp., of $P_G = n\Delta_{[m]}$). Each point $b \in P_G^-$ corresponds to a tree $T_{\mathbb{T}}(b) \in \mathcal{T}(\tau)$ with $\mathrm{LD}^-(T_{\mathbb{T}}(b)) = b$ so that the outgoing arrows of b in $\mathbb{T}$ in the direction of e_i are labeled by the neighbors of i in $T_{\mathbb{T}}(b)$.

Since all top-dimensional simplices Δ_T have the same volume (by [Pos09, Lemma 12.5]), it follows that a triangulation τ of Q_G corresponds to a maximal by size collection $\mathcal{T}(\tau)$ of pairwise compatible spanning trees of G.

Given a spanning tree $T \subset G$, introduce the *left-degree vector* $\mathrm{LD}^-(T) := (d_1, \dots, d_m)$ given by $d_i := \deg_i(T) - 1$. We similarly define the *right-degree vector* $\mathrm{RD}^-(T) := (d_{\bar{1}}, \dots, d_{\bar{n}})$.

Lemma 2.5 ([Pos09, Lemma 12.7])**.** *Given a triangulation τ of Q_G, we have $\mathrm{LD}^-(T) \neq \mathrm{LD}^-(T')$ for $T \neq T' \in \mathcal{T}(\tau)$, and the set $\{\mathrm{LD}^-(T) \mid T \in \mathcal{T}(\tau)\}$ equals $P_G^- \cap \mathbb{Z}^m$.*

In other words, every integer point of P_G^- appears as a left-degree vector for a unique tree in any triangulation. Thus a triangulation τ gives rise to a bijection $\phi_\tau : P_G^- \cap \mathbb{Z}^m \to P_{G^*}^- \cap Z^n$, where $G^* \subset K_{n,m}$ is the graph with edge set $\{(j, \bar{i}) \mid (i, \bar{j}) \in G\}$. In particular, when $G = K_{m,n}$, the map ϕ_τ is a bijection between $(n-1)\Delta^{m-1} \cap \mathbb{Z}^m$ and $(m-1)\Delta^{n-1} \cap \mathbb{Z}^n$. Note that each of the two sets has cardinality $\binom{n+m-2}{n-1}$.

3. Main results: the case $G = K_{m,n}$

We concentrate on characterizing triangulations by a set of axioms. For simplicity, we first state our definitions and results in the case when G is the complete bipartite graph $K_{m,n}$. For the rest of this section, we assume $G = K_{m,n}$. We denote $\Delta^{\mathbb{Z}}(m,k) := k\Delta_{[m]} \cap \mathbb{Z}^m$. Thus we have $P_G \cap \mathbb{Z}^m = \Delta^{\mathbb{Z}}(m,n)$ and $P_G^- \cap \mathbb{Z}^m = \Delta^{\mathbb{Z}}(m, n-1)$.

Let us consider a directed graph Γ with vertex set $V(\Gamma) = \Delta^{\mathbb{Z}}(m, n-1) \sqcup \Delta^{\mathbb{Z}}(m,n)$ and edge set $E(\Gamma) := \{b \to b + e_i \mid b \in \Delta^{\mathbb{Z}}(m, n-1), i \in [m]\}$. We alternatively denote an edge $b \to a$, where $a = b + e_i$ for some $i \in [m]$, by either $b \underset{i}{\to} \bullet$ or $\circ \underset{i}{\to} a$.

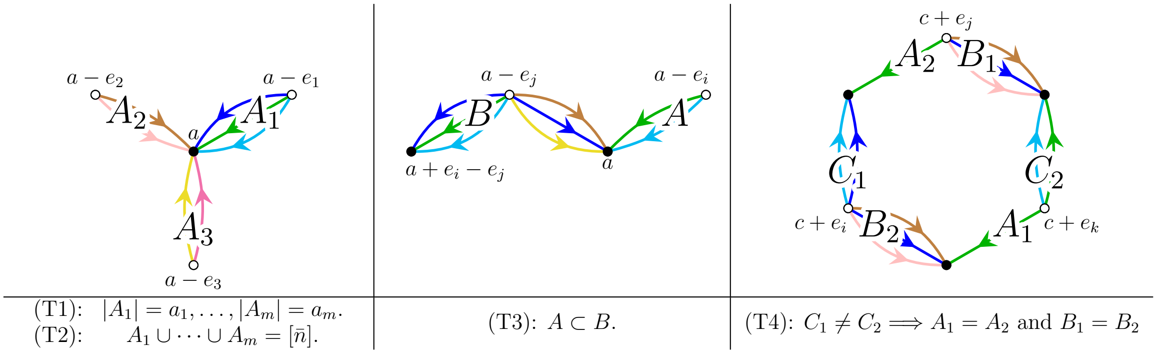

FIGURE 2. Axioms for trianguloids.

Definition 3.1. A *pre-trianguloid* is a map $\mathbb{T} : E(\Gamma) \to 2^{[\bar{n}]}$ satisfying the following axioms:

(T1) for every edge $(\circ \underset{i}{\to} a) \in E(\Gamma)$, we have $|\mathbb{T}(\circ \underset{i}{\to} a)| = a_i$.

(T2) for each $a \in \Delta^{\mathbb{Z}}(m,n)$ and $\bar{j} \in [\bar{n}]$, there exists an index $i \in [m]$ such that $\bar{j} \in \mathbb{T}(\circ \underset{i}{\to} a)$.

(T3) If both a and $a' := a + e_i - e_j$ belong to $\Delta^{\mathbb{Z}}(m,n)$ then we have
$$\mathbb{T}(\circ \underset{i}{\to} a) \subset \mathbb{T}(\circ \underset{i}{\to} a').$$

These axioms are illustrated in Figure 2.

Remark 3.2. By Axiom (T1), we have $\sum_{i\in[m]} |\mathbb{T}(\circ \underset{i}{\to} a)| = n$, and by Axiom (T2), the union of these sets is $[\bar{n}]$. Thus these sets are pairwise disjoint, so the index $i \in [m]$ in Axiom (T2) not only exists but is also unique.

Consider a triangulation τ of $Q_{K_{m,n}} = \Delta_{[m]} \times \Delta_{[\bar{n}]}$. By Lemma 2.5, for each $b \in \Delta^{\mathbb{Z}}(m, n-1)$, there is a unique tree $T_\tau(b) \in \mathcal{T}(\tau)$ such that $\mathrm{LD}^-(T_\tau(b)) = b$. Define a map $\mathbb{T}_\tau : E(\Gamma) \to 2^{[\bar{n}]}$ by

$$\mathbb{T}_\tau(b \underset{i}{\to} \bullet) := \{\bar{j} \mid (i,\bar{j}) \in T_\tau(b)\}. \tag{3.1}$$

Proposition 3.3. *If τ is a triangulation of $\Delta_{[m]} \times \Delta_{[\bar{n}]}$ then $\mathbb{T}_\tau$ is a pre-trianguloid.*

See Figure 1 for an example.

Conversely, given a pre-trianguloid $\mathbb{T} : E(\Gamma) \to 2^{[\bar{n}]}$ and a point $b \in \Delta^{\mathbb{Z}}(m, n-1)$, one can define a subgraph $T_{\mathbb{T}}(b) \subset K_{m,n}$ with edge set

$$T_{\mathbb{T}}(b) := \{(i,\bar{j}) \mid \bar{j} \in \mathbb{T}(b \underset{i}{\to} \bullet)\}. \tag{3.2}$$

Proposition 3.4. *For a pre-trianguloid $\mathbb{T}$ and a point $b \in \Delta^{\mathbb{Z}}(m, n-1)$, $T_{\mathbb{T}}(b)$ is a spanning tree of $K_{m,n}$.*

Thus $\{\Delta_{T_{\mathbb{T}}(b)} \mid b \in \Delta^{\mathbb{Z}}(m, n-1)\}$ is a collection of full-dimensional simplices whose total volume is equal to the volume of $\Delta_{[m]} \times \Delta_{[\bar{n}]}$. However, it may happen that these simplices do not in fact form a triangulation of $\Delta_{[m]} \times \Delta_{[\bar{n}]}$, see Figure 3 for an example.

We fix this by introducing an additional axiom.

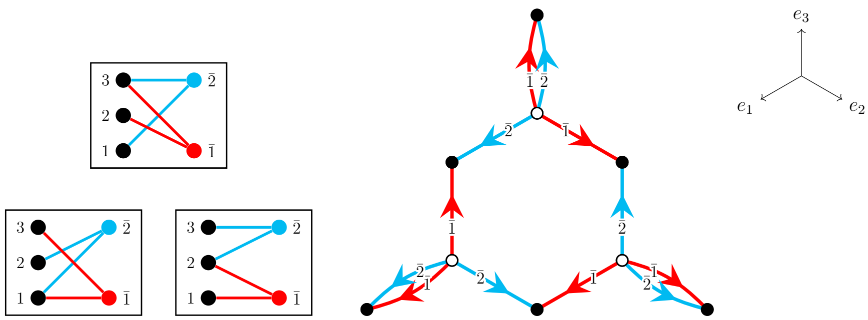

FIGURE 3. A pre-trianguloid for $m = 3, n = 2$ which does not correspond to a triangulation of $\Delta_{[m]} \times \Delta_{[\bar{n}]}$. The trees on the left are pairwise non-compatible.

Definition 3.5. A *trianguloid* is a pre-trianguloid $\mathbb{T} : E(\Gamma) \to 2^{[\bar{n}]}$ satisfying the following *Hexagon axiom*:

(T4) let $c \in \Delta^{\mathbb{Z}}(m, n-2)$ and consider three distinct indices $i, j, k \in [m]$ such that $\mathbb{T}(c + e_i \underset{j}{\to} \bullet) \neq \mathbb{T}(c + e_k \underset{j}{\to} \bullet)$. Then we have

$$\mathbb{T}(c + e_i \underset{k}{\to} \bullet) = \mathbb{T}(c + e_j \underset{k}{\to} \bullet) \quad \text{and} \quad \mathbb{T}(c + e_j \underset{i}{\to} \bullet) = \mathbb{T}(c + e_k \underset{i}{\to} \bullet).$$

Axioms (T1)-(T4) are illustrated in Figure 2.

The following is our main result for the case $G = K_{m,n}$.

Theorem 3.6. *The map $\tau \mapsto \mathbb{T}_\tau$ is a bijection between triangulations of $\Delta_{[m]} \times \Delta_{[\bar{n}]}$ and trianguloids.*

The generalization of this to arbitrary G is given in Theorem 5.6.

We now describe a compact way of encoding a pre-trianguloid. Introduce another directed graph Λ with vertex set $V(\Lambda) := \Delta^{\mathbb{Z}}(m, n-1)$ and edge set

$$E(\Lambda) := \{b \to b' \mid b, b' \in \Delta^{\mathbb{Z}}(m, n-1), \quad b' = b + e_i - e_j \quad \text{for some } i \neq j \in [m]\}.$$

Consider an edge $b \to b' \in E(\Lambda)$. Then by Axioms (T1) and (T3), there is a single index $\bar{j}$ such that $\mathbb{T}(b' \underset{i}{\to} \bullet) = \mathbb{T}(b \underset{i}{\to} \bullet) \sqcup \{\bar{j}\}$ (disjoint union). Thus each pre-trianguloid $\mathbb{T}$ defines an edge coloring $\mathcal{E}_{\mathbb{T}} : E(\Lambda) \to [\bar{n}]$ sending an edge $(b \to b') \in E(\Lambda)$ to the above index $\bar{j}$. It is easy to see that a pre-trianguloid can in fact be uniquely reconstructed from this map. An example is given in Figure 4.

We finish by going back to our original question, deducing an analog of [BZ93, Theorem 5] as a simple consequence of the above results.

Theorem 3.7. *For two different triangulations τ, τ' of $Q_{K_{m,n}}$, the maps $\phi_\tau, \phi_{\tau'}$ are different as well.*

4. MOTIVATION

Before stating our main results for the case of arbitrary connected $G \subset K_{m,n}$, we discuss (very informally) some of the objects corresponding to triangulations of $\Delta_{[m]} \times$

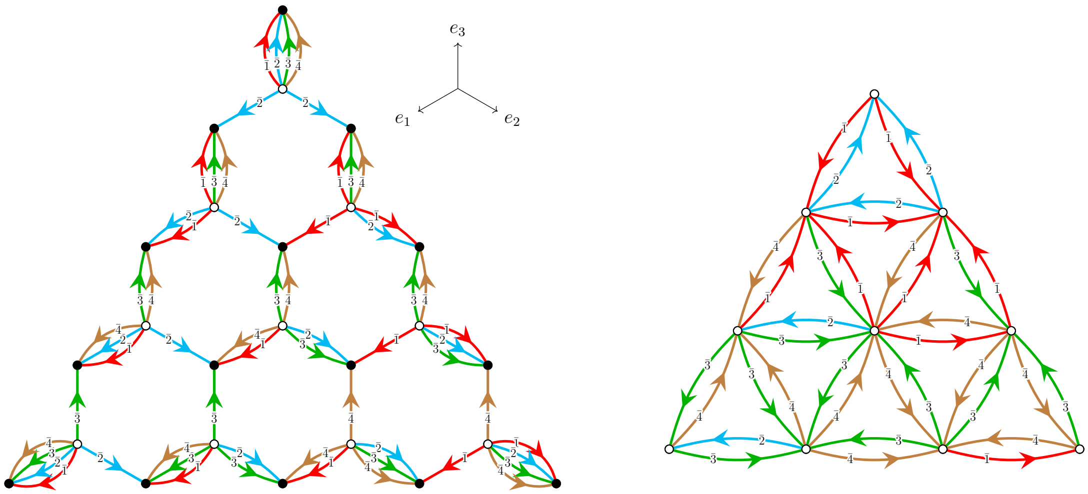

FIGURE 4. A trianguloid $\mathbb{T}$ (left) and the corresponding edge coloring $\mathcal{E}_{\mathbb{T}}$ (right). Here $m = 3$ and $n = 4$.

$\Delta_{[\bar{n}]}$ that have been studied earlier. The main goal of this section is to provide intuition and motivating examples; the only two things that we actually use in the remainder of the paper are Definition 4.1 and part (3) of Lemma 4.3.

4.1. **Forests, matchings, and tropical oriented matroids.** In this section, we list several ways to describe a triangulation τ of Q_G, where $G \subset K_{m,n}$ is an arbitrary connected graph. Recall that $\mathcal{F}(\tau)$ and $\mathcal{T}(\tau)$ denotes the collection of *forests* and *trees* of τ respectively.

Definition 4.1. We say that a forest $F \subset G$ is a *right semi-matching* if $\deg_{\bar{j}}(F) = 1$ for all $\bar{j} \in [\bar{n}]$, and denote

$$\mathcal{RSM}(\tau) = \{F \in \mathcal{F}(\tau) \mid F \text{ is a right semi-matching}\}.$$

We similarly define *left semi-matchings* to be forests $F \subset G$ such that $\deg_i(F) = 1$ for all $i \in [m]$, and denote by $\mathcal{LSM}(\tau)$ the set of left semi-matchings in $\mathcal{F}(\tau)$. Finally, recall that a forest F is a *partial matching* if $0 \leq \deg_i(F), \deg_{\bar{j}}(F) \leq 1$ for all $i \in [m]$ and $\bar{j} \in [\bar{n}]$. In this case, we call the set $\mathrm{I}(F) := \{i \in [m] \mid \deg_i(F) = 1\}$ (resp., $\mathrm{J}(F) := \{\bar{j} \in [\bar{n}] \mid \deg_{\bar{j}}(F) = 1\}$) the *left support* (resp., the *right support*) of F, and say that F is a *matching between* $\mathrm{I}(F)$ *and* $\mathrm{J}(F)$.

We denote

$$\mathcal{PM}(\tau) = \{F \in \mathcal{F}(\tau) \mid F \text{ is a partial matching}\}.$$

The following result (cf. Figure 5) will follow as a simple corollary to Lemma 6.1.

Proposition 4.2. *A triangulation τ of Q_G is determined uniquely by each of the following sets:*

- *$\mathcal{T}(\tau)$;*
- *$\mathcal{RSM}(\tau)$;*
- *$\mathcal{LSM}(\tau)$;*

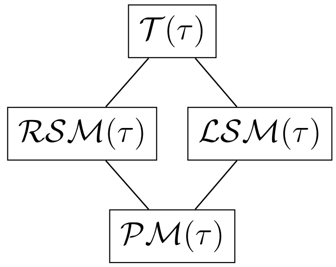

FIGURE 5. Different collections of forests of τ that determine it.

- *$\mathcal{PM}(\tau)$.*

More precisely, for each of the four collections above, $\mathcal{F}(\tau)$ is equal to the set of all forests $F \subset G$ compatible (see Definition 2.3) with all F' belonging to that collection.

Proof. Clearly τ is determined by $\mathcal{T}(\tau)$, and $\mathcal{T}(\tau)$ determines $\mathcal{RSM}(\tau)$. Let us show that $\mathcal{RSM}(\tau)$ determines $\mathcal{PM}(\tau)$. Each partial matching $M \in \mathcal{PM}(\tau)$ is contained inside some tree $T \in \mathcal{T}(\tau)$ because τ is a pure simplicial complex, and then it is easy to see that there exists a right semi-matching $F \subset T$ such that $M \subset F$. This implies that $F \in \mathcal{RSM}(\tau)$. Thus $\mathcal{PM}(\tau)$ is the set of all partial matchings of G that are contained in some element of $\mathcal{RSM}(\tau)$, so $\mathcal{RSM}(\tau)$ determines $\mathcal{PM}(\tau)$. The proof that $\mathcal{T}(\tau)$ determines $\mathcal{LSM}(\tau)$ which determines $\mathcal{PM}(\tau)$ is completely analogous. It suffices to show that $\mathcal{PM}(\tau)$ determines τ. Explicitly, $\mathcal{F}(\tau)$ is the collection of all forests $F \subset G$ that do not contain a partial matching that is not in $\mathcal{PM}(\tau)$. This fact follows from Lemma 2.4 (whose proof we defer to Section 6). $\square$

We now review the relationship between the above objects and *tropical oriented matroids* of [AD09]. It was conjectured in [AD09, Conjecture 5.1] that tropical oriented matroids are in bijection with subdivisions of $\Delta_{[m]} \times \Delta_{[\bar{n}]}$. Oh and Yoo [OY11] proved that *generic* tropical oriented matroids are in bijection with triangulations of $\Delta_{[m]} \times \Delta_{[\bar{n}]}$, and the case of general subdivisions was completed by Horn [Hor16].

A tropical oriented matroid M is by definition a collection of *types* satisfying some axioms, see [AD09]. In the language of triangulations, types correspond to forests $F \in \mathcal{F}(\tau)$ such that $\deg_{\bar{j}}(F) \geq 1$ for all $\bar{j} \in [\bar{n}]$. Alternatively, a tropical oriented matroid is determined by the collection of its *topes* or by the collection of its *vertices*, see [AD09, Theorems 4.4 and 4.6]. The topes correspond to the elements of $\mathcal{RSM}(\tau)$ and the vertices correspond to the elements of $\mathcal{T}(\tau)$, i.e., the right semi-matchings and the trees of τ, respectively. Thus in the case of $G = K_{m,n}$, Proposition 4.2 follows from [AD09, Theorems 4.4 and 4.6] together with [OY15, Lemma 4.5].

For a forest $F \subset G$, define

$$\mathrm{LD}(F) := (\deg_i(F))_{i \in [m]} \in \mathbb{Z}^m, \quad \mathrm{RD}(F) := (\deg_{\bar{j}}(F))_{\bar{j} \in [\bar{n}]} \in \mathbb{Z}^n.$$

Let us denote by $\mathcal{PM}(G)$ the set of all partial matchings F such that $F \subset G$. We define $\mathrm{IJ}_G \subset 2^{[m]} \times 2^{[\bar{n}]}$ by

$$\mathrm{IJ}_G := \{(\mathrm{I}(F), \mathrm{J}(F)) \mid F \in \mathcal{PM}(G)\}.$$

Identifying a pair $(I,J) \in 2^{[m]} \times 2^{[\bar{n}]}$ with a vector $e_I + e_J := \sum_{i\in I} e_i + \sum_{\bar{j}\in J} e_{\bar{j}} \in \mathbb{R}^{m+n}$, we see that IJ_G is the set of lattice points of a certain polytope in $\mathbb{R}^{m+n}$ which we call the *matching support polytope* MSP_G of G:

$$\mathrm{MSP}_G := \mathrm{Conv}\left(\{e_I + e_J \mid (I,J) \in \mathrm{IJ}_G\}\right) = \mathrm{Conv}\left(\{(\mathrm{LD}(F), \mathrm{RD}(F)) \mid F \in \mathcal{PM}(G)\}\right).$$

We prove the following generalization of Lemma 2.5.

Lemma 4.3. *Given a triangulation τ of Q_G, the following maps are bijections:*
(1) $\mathrm{LD}^- : \mathcal{T}(\tau) \to P_G^- \cap \mathbb{Z}^m$;
(2) $\mathrm{RD}^- : \mathcal{T}(\tau) \to P_{G^*}^- \cap \mathbb{Z}^n$;
(3) $\mathrm{LD} : \mathcal{RSM}(\tau) \to P_G \cap \mathbb{Z}^m$;
(4) $\mathrm{RD} : \mathcal{LSM}(\tau) \to P_{G^*} \cap \mathbb{Z}^n$;
(5) $(\mathrm{LD}, \mathrm{RD}) : \mathcal{PM}(\tau) \to \mathrm{MSP}_G \cap \mathbb{Z}^{m+n}$.

4.2. **Newton polytopes and products of minors.** Fix a connected graph $G \subset K_{m,n}$ and consider an $m \times n$ matrix $f_G = (f_{ij})$ with f_{ij} being an indeterminate for $(i,\bar{j}) \in G$ and $f_{ij} = 0$ otherwise.

For two subsets $I \subset [m]$, $J \subset [\bar{n}]$ of the same size, define $\Delta_{I,J}(f_G)$ to be the minor of f_G with row set I and column set J. Thus $\Delta_{I,J}(f_G)$ is a nonzero polynomial if and only if $(I,J) \in \mathrm{IJ}_G$. Let N_G be the *Newton polytope* of the product of all non-zero minors of f_G:

$$N_G := \mathrm{Newton}\left(\prod_{(I,J)\in \mathrm{IJ}_G} \Delta_{I,J}(f_G)\right) \subset \mathbb{R}^G. \tag{4.1}$$

For the case $G = K_{m,n}$ it was shown in [GKZ08, Example 10.C.1.3(b)] that N_G is the *secondary polytope* of $\Delta_{[m]} \times \Delta_{[\bar{n}]}$. We generalize this statement to arbitrary G:

Proposition 4.4. *For $G \subset K_{m,n}$, N_G is combinatorially equivalent to the secondary polytope of Q_G. More precisely, these polytopes have the same normal fans.*

Proof. Let us consider an $m \times n$ matrix $h = (h_{ij}) \in \mathbb{R}^G$ such that $h_{ij} = 0$ when $(i,\bar{j}) \notin G$. It defines a *regular subdivision* τ_h of Q_G as follows. Let $e_0, e_1, \dots, e_m, e_{\bar{1}}, \dots, e_{\bar{n}}$ be a basis of $\mathbb{R}^{m+n+1}$, and consider a polytope $Q_G(h) \subset \mathbb{R}^{m+n+1}$ defined as the convex hull of $e_i + e_{\bar{j}} + h_{ij}e_0$ for all $(i,\bar{j}) \in G$. Then τ_h is the subdivision of Q_G obtained by projecting the lower faces of $Q_G(h)$ from $\mathbb{R}^{m+n+1}$ to $\mathbb{R}^{m+n}$. It is easy to see that Δ_F is contained in a face of τ_h for some forest $F \subset G$ if and only if for each partial matching $M \subset F$ and any other matching $M' \subset G$ with $\mathrm{I}(M) = \mathrm{I}(M')$, $\mathrm{J}(M) = \mathrm{J}(M')$, we have

$$\sum_{(i,\bar{j})\in M} h_{ij} \leq \sum_{(i,\bar{j})\in M'} h_{ij}. \tag{4.2}$$

If we fix $I := \mathrm{I}(M)$ and $J := \mathrm{J}(M)$ then the set of all inequalities of the form (4.2) describes exactly the normal fan of $\mathrm{Newton}(\Delta_{I,J}(f_G))$. Thus the normal fan of the secondary polytope of Q_G is their common refinement. On the other hand, N_G is the Minkowski sum of $\mathrm{Newton}(\Delta_{I,J}(f_G))$ over all $(I,J) \in \mathrm{IJ}_G$, and thus its normal fan is the common refinement of the normal fans of the summands. □

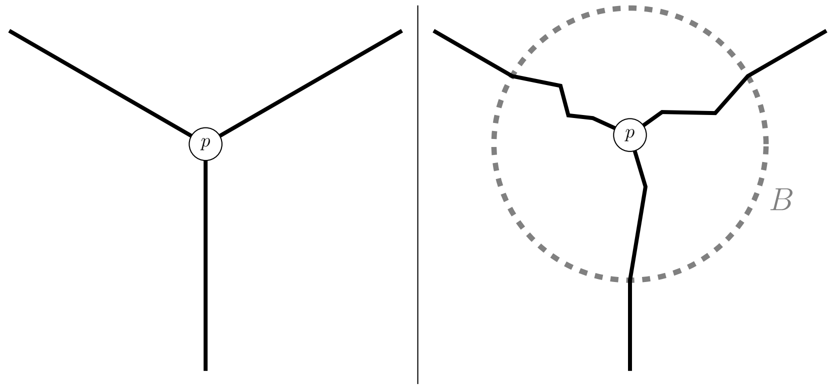

FIGURE 6. A tropical line (left) and a tropical pseudoline (right).

The vertices of the secondary polytope of Q_G correspond to *coherent* (or *regular*) triangulations of Q_G. Proposition 4.4 implies that each such triangulation corresponds to a vertex of N_G. Note that N_G is a Minkowski sum of Newton polytopes of $\Delta_{I,J}(f_G)$, thus a vertex of N_G corresponds to choosing a vertex inside each summand, that is, for each pair $(I, J) \in \mathrm{IJ}_G$, we choose a single matching $M \subset G$ with $\mathrm{I}(M) = I$ and $\mathrm{J}(M) = J$. Note that by Lemma 4.3, part (5), any triangulation of Q_G (not necessarily a coherent one) corresponds to a choice of a single matching M for each pair $(I, J) \in \mathrm{IJ}_G$. In the case when $G = K_{m,n}$ is the complete bipartite graph, it was shown in [OY15] that any such collection of matchings satisfying a certain set of axioms (most notably, the *linkage axiom* of [SZ93, BZ93]) equals $\mathcal{PM}(\tau)$ for some triangulation τ of $\Delta_{[m]} \times \Delta_{[\bar{n}]}$. In [SZ93, BZ93], the authors considered a closely related object, namely the Newton polytope of the product of *maximal* minors (as opposed to *all* minors as we did in (4.1)) of $f_{K_{m,n}}$. It would be interesting to generalize the constructions of [OY15, SZ93, BZ93] to arbitrary subgraphs $G \subset K_{m,n}$.

4.3. **Lozenge tilings and tropical pseudoline arrangements.** Throughout this section, we assume that $m = 3$. We refer the reader to Figure 7 for some of the bijections that we mention below.

Our first goal is to recast the notion of a *tropical pseudohyperplane* for $m = 3$ in elementary terms. See [DS04, AD09] for precise definitions for general m.

Suppose we are given three unit vectors u_1, u_2, and u_3 in $\mathbb{R}^2$ with $u_1 + u_2 + u_3 = 0$, and let $B \subset \mathbb{R}^2$ be the unit ball centered at the origin. Given a point $p \in B$, a *tropical line* L centered at p is a union of three rays $r_1, r_2, r_3 : \mathbb{R}_{\geq 0} \to \mathbb{R}^2$ such that for each $i = 1, 2, 3$, we have $r_i(t) = p - tu_i$ for all $t \geq 0$. A *tropical pseudoline* L is an image of a tropical line under a piecewise-linear homeomorphism ϕ of $\mathbb{R}^2$ that fixes $\mathbb{R}^2 \setminus B$. The image $\phi(p)$ is called the *center* of L and the piecewise-linear curves $\phi \circ r_i$ are called the *legs* of L. See Figure 6.

We say that a family $L^{(\bar{1})}, L^{(\bar{2})}, \dots, L^{(\bar{n})}$ of tropical pseudolines form an *arrangement* if any two of them intersect exactly once (and this intersection is transversal), the center of $L^{(\bar{j})}$ is not contained in $L^{(\bar{k})}$ for $\bar{j} \neq \bar{k}$, and no three of them intersect simultaneously

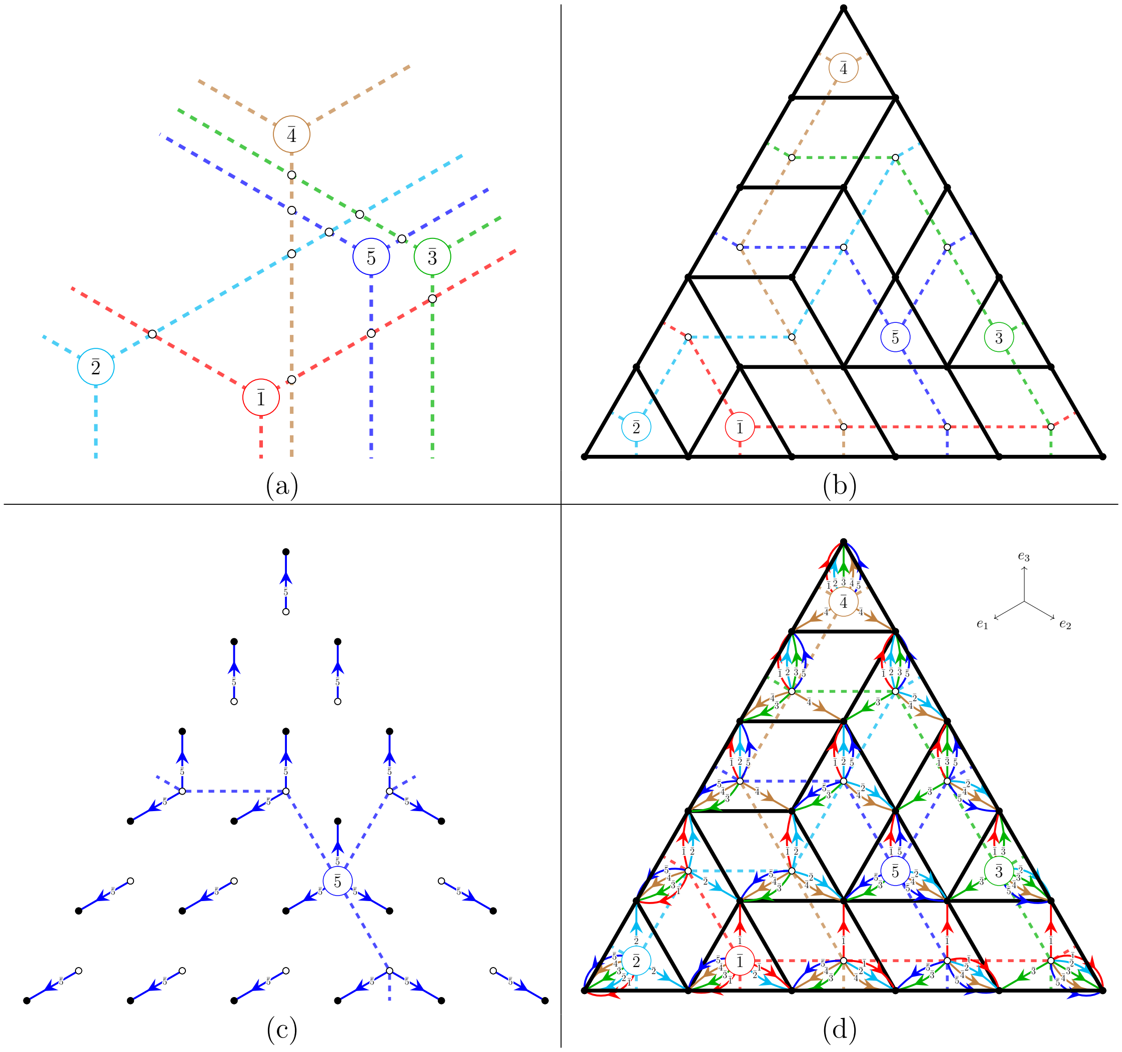

FIGURE 7. The case $m = 3$, $n = 5$. (a) A tropical pseudoline arrangement. (b) The corresponding lozenge tiling of $n\Delta_{[m]}$. (c) The pseudoline $\bar{5}$ together with the arrows $b \to a$ of $E(\Gamma)$ for which $\bar{5} \in \mathbb{T}(b \to a)$. (d) A tropical pseudoline arrangement, a lozenge tiling, and a trianguloid, all corresponding to each other.

at a single point. An arrangement of 5 tropical pseudolines is shown in Figure 7 (a). In this case, all of them are actual tropical lines.

Remark 4.5. We note that any arrangement of tropical lines yields a (very degenerate) *honeycomb* in the sense of Knutson-Tao [KT99]. Such special honeycombs provide a simple proof of the *weak PRV conjecture* [PRRV67], which was proven in full generality in [Kum88, Mat89, Pol89]. We refer the reader to [KT99, Section 4] for details.

Since each pair of tropical pseudolines in an arrangement must intersect exactly once, there are $\binom{n}{2}$ points of intersection between them. Together with the n centers, these

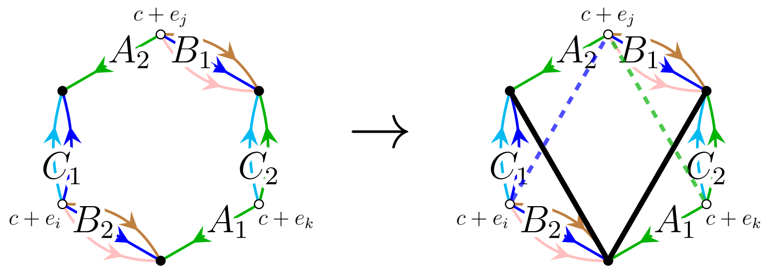

FIGURE 8. Reconstructing a lozenge tiling and a tropical pseudoline arrangement from a trianguloid. If $C_1 \neq C_2$ then the segments of the lozenge tiling connect $c+e_i+e_k$ to $c+e_i+e_j$ and to $c+e_k+e_j$. The points $c+e_i$ and $c+e_j$ are connected by a segment (dashed) of the tropical pseudoline $L^{(\bar{x})}$, where $\{\bar{x}\} = C_1 \setminus C_2$. Similarly, the points $c+e_k$ and $c+e_j$ are connected by a segment of the tropical pseudoline $L^{(\bar{y})}$, where $\{\bar{y}\} = C_2 \setminus C_1$.

$\binom{n+1}{2}$ points can be uniquely mapped to the points in $\Delta^{\mathbb{Z}}(3, n-1)$ so that whenever two of them belong to a leg $r_i^{(\bar{j})}(t)$ of $L^{(\bar{j})}$, the one that is closer to the center of $L^{(\bar{j})}$ maps to a point in $\Delta^{\mathbb{Z}}(3, n-1)$ with a larger i-th coordinate. See Figure 7 (b).

In fact, this gives a simple bijection between arrangements of n tropical pseudolines and lozenge tilings of a *holey triangle*. In the above setting, let T_n be the convex hull of nu_1, nu_2, nu_3. A *lozenge tiling* of T_n is a subdivision of T_n into $\binom{n}{2}$ lozenges and n *upright triangles*. Here an upright triangle is the convex hull of u_1, u_2, u_3 (possibly shifted by some vector) and a *lozenge* is a union of an upright triangle and its reflection about one of its sides. The lozenge tiling of T_5 corresponding to the above arrangement of 5 tropical pseudolines is shown in solid black lines in Figure 7 (b).

Tropical pseudoline arrangements and lozenge tilings of holey triangles are special cases for $m = 3$ of *tropical pseudohyperplane arrangements* [AD09, Hor16] and *fine mixed subdivisions of* $n\Delta_{[m]}$ [San05, HRS00], respectively.

Let us now explain a way of constructing a trianguloid from these objects. Fix $\bar{j} \in [\bar{n}]$ and consider the image of $L^{(\bar{j})}$ in $(n-1)\Delta_{[3]}$, see Figure 7 (c). Its complement in $(n-1)\Delta_{[3]}$ consists of three connected components. For $i = 1, 2, 3$, we denote by $C_i^{(\bar{j})} \subset (n-1)\Delta_{[3]}$ the closure of the connected component that contains the vertex $(n-1)e_i$. Each point of $\Delta^{\mathbb{Z}}(3, n-1)$ now belongs to $C_i^{(\bar{j})}$ for one or several values of i. We then define a trianguloid $\mathbb{T}$ by the condition that $\bar{j}$ belongs to $\mathbb{T}(b \underset{i}{\to} \bullet)$ for an edge $(b \underset{i}{\to} \bullet) \in E(\Gamma)$ whenever b belongs to $C_i^{(\bar{j})}$. Thus for example if b is the image of the center of $L^{(\bar{j})}$ then it belongs to $C_i^{(\bar{j})}$ for all $i = 1, 2, 3$. It is easy to see that thus defined map $\mathbb{T} : E(\Gamma) \to 2^{[\bar{n}]}$ is indeed a trianguloid, and the corresponding collection of spanning trees of $K_{3,n}$ yields a fine mixed subdivision of $n\Delta_{[3]}$ that coincides with the lozenge tiling of T_n described above.

A way of describing the inverse correspondence can be given using Axiom (T4). Namely let $c \in \Delta^{\mathbb{Z}}(3, n-2)$ be a point and let i, j, k be three indices with $\{i, j, k\} =$

$\{1,2,3\}$ so that

$$\mathbb{T}(c+e_i \underset{k}{\to} \bullet) = \mathbb{T}(c+e_j \underset{k}{\to} \bullet) \quad \text{and} \quad \mathbb{T}(c+e_j \underset{i}{\to} \bullet) = \mathbb{T}(c+e_k \underset{i}{\to} \bullet).$$

Then it is easy to see that we in fact must have $\mathbb{T}(c+e_i \underset{j}{\to} \bullet) \neq \mathbb{T}(c+e_k \underset{j}{\to} \bullet)$, i.e., the converse to Axiom (T4) holds for $m = 3$. Indeed, otherwise the three sets $\mathbb{T}(c+e_i \underset{j}{\to} \bullet) = \mathbb{T}(c+e_k \underset{j}{\to} \bullet)$, $\mathbb{T}(c+e_i \underset{k}{\to} \bullet) = \mathbb{T}(c+e_j \underset{k}{\to} \bullet)$, $\mathbb{T}(c+e_j \underset{i}{\to} \bullet) = \mathbb{T}(c+e_k \underset{i}{\to} \bullet)$ would be pairwise disjoint (see Remark 3.2), so their union would have size $c_i + c_j + c_k + 3 = n+1$ which is impossible for a subset of $[\bar{n}]$. Now, let us connect $c+e_i+e_k$ with $c+e_i+e_j$ and with $c+e_j+e_k$ using solid black lines, see Figure 8. We claim that the union of these solid black lines over all hexagons, together with the boundary of $n\Delta_{[3]}$, yields the lozenge tiling of a holey triangle corresponding to $\tau_{\mathbb{T}}$. Similarly, denote $\bar{x}$ to be the unique element of $\mathbb{T}(c+e_i \underset{j}{\to} \bullet) \setminus \mathbb{T}(c+e_k \underset{j}{\to} \bullet)$ and $\bar{y}$ to be the unique element of $\mathbb{T}(c+e_k \underset{j}{\to} \bullet) \setminus \mathbb{T}(c+e_i \underset{j}{\to} \bullet)$. Then connect $c+e_i$ with $c+e_j$ using a dashed line labeled $\bar{x}$ and connect $c+e_k$ with $c+e_j$ using a dashed line labeled $\bar{y}$, as in Figure 8. We claim that the union of these dashed lines over all hexagons yields the tropical pseudoline arrangement that corresponds to $\mathbb{T}$. We encourage the reader to examine the hexagons of this form in Figure 7 (d), which is the superposition of a trianguloid, a tropical pseudoline arrangement, and a lozenge tiling, all corresponding to the same triangulation of $\Delta_{[3]} \times \Delta_{[\bar{5}]}$.

5. Main results: the case of arbitrary G

We extend the results of Section 3 to arbitrary connected subgraphs $G \subset K_{m,n}$.

Define a directed graph Γ_G with vertex set $(P_G \sqcup P_G^{\pm}) \cap \mathbb{Z}^m$ and edge set $E(\Gamma_G) := \{a - e_i \to a \mid a \in P_G \cap \mathbb{Z}^m, i \in [m] : a_i > 0\}$. Note that by the definition of $P_G^{\pm}$, we have $a - e_i \in P_G^{\pm} \cap \mathbb{Z}^m$. We again abbreviate the edge $a - e_i \to a$ as either $a - e_i \underset{i}{\to} \bullet$ or $\circ \underset{i}{\to} a$.

Definition 5.1. A *pre-trianguloid* is a map $\mathbb{T} : E(\Gamma_G) \to 2^{[\bar{n}]}$ satisfying the following axioms:

- (T1') for every edge $(\circ \underset{i}{\to} a) \in E(\Gamma_G)$, we have $|\mathbb{T}(\circ \underset{i}{\to} a)| = a_i$.
- (T2') for each $a \in P_G \cap \mathbb{Z}^m$ and $\bar{j} \in [\bar{n}]$, there exists an index $i \in N_{\bar{j}}(G)$ such that $\bar{j} \in \mathbb{T}(\circ \underset{i}{\to} a)$.
- (T3') If both a and $a' := a + e_i - e_j$ belong to $P_G \cap \mathbb{Z}^m$ and $a_i > 0$ then

$$\mathbb{T}(\circ \underset{i}{\to} a) \subset \mathbb{T}(\circ \underset{i}{\to} a').$$

It is clear that Remark 3.2 generalizes to the case of arbitrary G. We also note that if $a_i = 0$ for some $a \in P_G \cap \mathbb{Z}^m$ and $i \in [m]$ then there is no edge $\circ \underset{i}{\to} a$ in $E(\Gamma_G)$ because $a - e_i \notin P_G^{\pm}$. However, in this case Axiom (T1') would require $\mathbb{T}(\circ \underset{i}{\to} a)$ to have zero cardinality, and in fact setting $\mathbb{T}(\circ \underset{i}{\to} a) := \emptyset$ for all such pairs of a and i does not have any effect on our arguments.

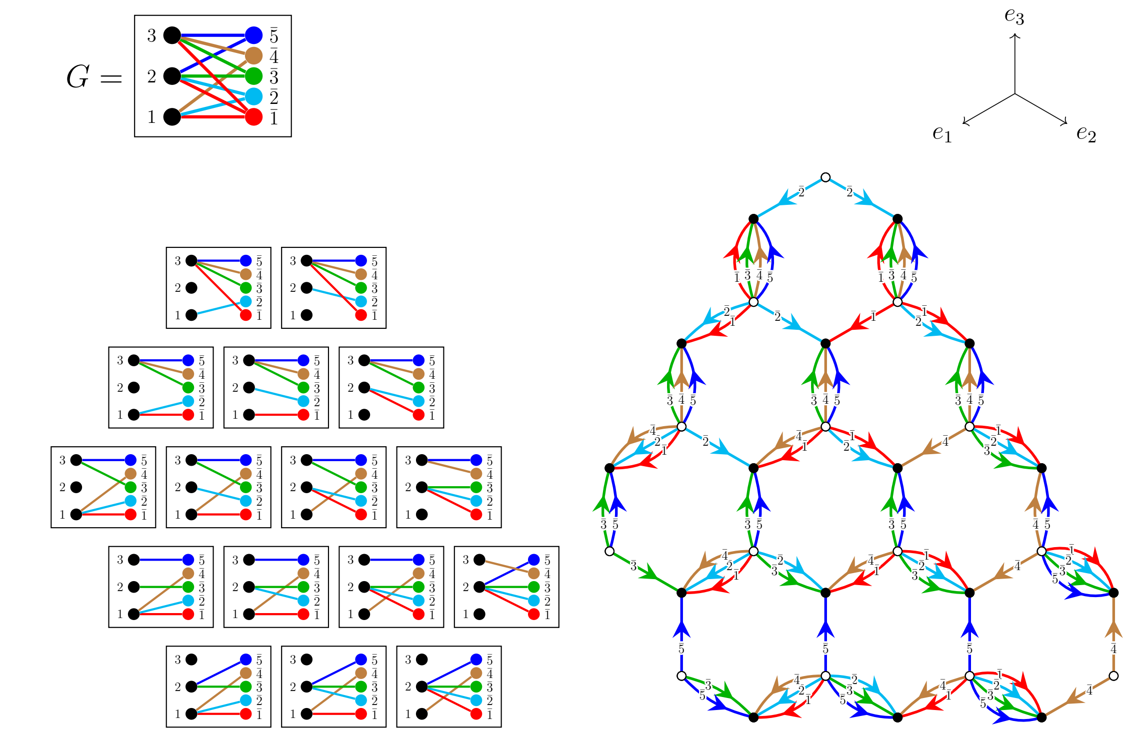

FIGURE 9. A graph $G \subset K_{m,n}$ for $m = 3, n = 5$ and a collection $\mathcal{RSM}(\tau)$ of right semi-matchings for some triangulation τ of Q_G (left). The corresponding trianguloid $\mathbb{T}_\tau$ (right). Each black vertex $a \in P_G \cap \mathbb{Z}^m$ of $\mathbb{T}$ corresponds to a unique right semi-matching $F_\tau(a) \in \mathcal{RSM}(\tau)$ such that $\mathrm{LD}(F) = a$. In this case, the incoming arrows of a in $\mathbb{T}$ in the direction of e_i are labeled by the neighbors of i in $F_\tau(a)$. The white vertices of $\mathbb{T}$ are the lattice points of $P_G^{\pm}$.

Similarly to the case $G = K_{m,n}$, for any triangulation τ of Q_G and any $b \in P_G^- \cap \mathbb{Z}^m$, there is a unique tree $T_\tau(b) \in \mathcal{T}(\tau)$ such that $\mathrm{LD}^-(T_\tau(b)) = b$, so we can define $\mathbb{T}_\tau(b \underset{i}{\to} \bullet)$ by (3.1). However, this does not define $\mathbb{T}_\tau$ on all $E(\Gamma_G)$ because some edges of Γ_G are of the form $b \to a$ for $b \in P_G^{\pm} \setminus P_G^-$. Instead, we use right semi-matchings from Definition 4.1.

Recall that by Lemma 4.3, part (3), LD is a bijection between the set $\mathcal{RSM}(\tau)$ of right semi-matchings of τ and the set $P_G \cap \mathbb{Z}^m$ of lattice points of P_G. Denote by $F_\tau : P_G \cap \mathbb{Z}^m \to \mathcal{RSM}(\tau)$ the inverse of this bijection. Given a triangulation τ of Q_G, define a map $\mathbb{T}_\tau : E(\Gamma_G) \to 2^{[\bar{n}]}$ by

$$\mathbb{T}_\tau(\circ \underset{i}{\to} a) = N_i(F_\tau(a)) \tag{5.1}$$

for all $a \in P_G \cap \mathbb{Z}^m$ and all $i \in [m]$ such that $a_i > 0$.

We have the analog of Proposition 3.3.

Proposition 5.2. *If τ is a triangulation of Q_G then $\mathbb{T}_\tau$ is a pre-trianguloid.*

It may seem that the definition (5.1) of $\mathbb{T}_\tau$ for the case of arbitrary G differs from the corresponding definition (3.1) for the case of $G = K_{m,n}$. The next lemma shows that this is not the case.

Lemma 5.3. *Let τ be a triangulation of Q_G and define $\mathbb{T} := \mathbb{T}_\tau$ by (5.1). For $b \in P_G^-$, the collection $T_{\mathbb{T}}(b)$ of edges given by (3.1) is the unique spanning tree of G satisfying $\mathrm{LD}^-(T_{\mathbb{T}}(b)) = b$ and $\Delta_{T_{\mathbb{T}}(b)} \in \tau$.*

Thus in the case $G = K_{m,n}$, the two definitions (3.1) and (5.1) of $\mathbb{T}_\tau$ agree with each other.

Proposition 5.4. *For a pre-trianguloid $\mathbb{T}$ and a point $b \in P_G^- \cap \mathbb{Z}^m$, $T_{\mathbb{T}}(b)$ is a spanning tree of G.*

To generalize the definition of a trianguloid to the case of an arbitrary G, we slightly modify Axiom (T4) for points on the boundary of P_G^-.

Definition 5.5. A *trianguloid* is a pre-trianguloid $\mathbb{T} : E(\Gamma_G) \to 2^{[\bar{n}]}$ satisfying the following *Hexagon axiom*:

(T4') let $c \in \mathbb{Z}^m$ and consider three distinct indices $i, j, k \in [m]$ such that $c+e_i, c+e_k \in P_G^-$ and $\mathbb{T}(c + e_i \underset{j}{\to} \bullet) \neq \mathbb{T}(c + e_k \underset{j}{\to} \bullet)$. Then we have $c + e_j \in P_G^-$ and

$$\mathbb{T}(c + e_i \underset{k}{\to} \bullet) = \mathbb{T}(c + e_j \underset{k}{\to} \bullet) \quad \text{and} \quad \mathbb{T}(c + e_j \underset{i}{\to} \bullet) = \mathbb{T}(c + e_k \underset{i}{\to} \bullet).$$

We are ready to state our main result:

Theorem 5.6. *The map $\tau \mapsto \mathbb{T}_\tau$ is a bijection between triangulations of Q_G and trianguloids.*

Theorem 3.7 also generalizes to the case of arbitrary G.

Theorem 5.7. *For two different triangulations τ, τ' of Q_G, the maps $\phi_\tau, \phi_{\tau'}$ are different as well.*

6. From triangulations to trianguloids

In this section, we show that for any triangulation τ of Q_G, the map $\mathbb{T}_\tau : E(\Gamma_G) \to 2^{[\bar{n}]}$ given by (5.1) is a trianguloid. We work in the generality of arbitrary connected $G \subset K_{m,n}$. Before we proceed, we need to show that the map F_τ used in (5.1) is well defined, thus we begin by showing Lemma 4.3.

First, we discuss the compatibility condition of [Pos09]. Given two forests $F, F' \subset G$, let $U(F, F')$ be a *directed* graph with edge set

$$\{i \to \bar{j} \mid (i, \bar{j}) \in F\} \cup \{\bar{j} \to i \mid (i, \bar{j}) \in F'\}.$$

The following result generalizes Lemma 2.4 and completes the proof of Proposition 4.2.

Lemma 6.1. *Given two forests $F, F' \subset G$, the following conditions are equivalent:*

(1) *the simplices Δ_F, $\Delta_{F'}$ intersect by their common face;*
(2) *the forests F and F' are compatible in the sense of Definition 2.3;*
(3) *$U(F, F')$ contains no directed cycles of length 3 or more.*

Proof. The equivalence of (1) and (3) is proven in [Pos09, Lemma 12.6] for the case when F and F' are spanning trees of G, but the proof translates verbatim to the case of forests. The fact that (3) implies (2) is obvious. Finally, note that if there exist $M \subset F$ and $M' \subset F'$ as in Definition 2.3, i.e., such that $\mathrm{I}(M) = \mathrm{I}(M')$ and $\mathrm{J}(M) = \mathrm{J}(M')$, then for every vertex of G, its indegree in $U(M, M')$ equals to its outdegree in $U(M, M')$, and thus $U(M, M')$ contains a directed cycle of length at least 4 because we have assumed $M \neq M'$. This shows that (2) implies (3), finishing the proof of the lemma. □

We need one more step before proving Lemma 4.3.

Lemma 6.2. *Suppose that τ is a triangulation of Q_G and consider two forests $F, F' \in \mathcal{RSM}(\tau)$. Then we have $\mathrm{LD}(F) \neq \mathrm{LD}(F')$ for $F \neq F'$.*

Proof. Since F and F' both belong to τ, they must be compatible by Lemma 6.1, but on the other hand, the assumptions of Lemma 6.2 force F and F' to satisfy $\deg_i(F) = \deg_i(F')$ and $\deg_{\bar{j}}(F) = \deg_{\bar{j}}(F')$ for all $i \in [m]$ and $\bar{j} \in [\bar{n}]$. Thus there is a directed cycle in $U(F, F')$, hence they are not compatible, a contradiction. □

Proof of Lemma 4.3. Parts (1) and (2) follow from Lemma 2.5.

We now prove part (3). Let τ be a triangulation of Q_G. By Lemma 6.2, we only need to show that for every lattice point $a \in P_G \cap \mathbb{Z}^m$, there exists a forest $F \in \mathcal{RSM}(\tau)$ such that $\mathrm{LD}(F) = a$. It is shown in [Pos09, Section 14] that τ corresponds to a *fine mixed subdivision* of P_G and it follows from the proof of [Pos09, Proposition 14.12] that a is a vertex of that mixed subdivision. It remains to note that such a vertex corresponds precisely to a simplex Δ_F for some $F \in \mathcal{RSM}(\tau)$. We are done with the proof of part (3). Part (4) is completely analogous.

Finally, we show part (5). Let $(I, J) \in \mathrm{IJ}_G$. By Lemma 6.1, there is at most one matching $F \in \mathcal{PM}(\tau)$ such that $\mathrm{I}(F) = I$ and $\mathrm{J}(F) = J$, and it remains to show that such a matching exists. Indeed, denote $k := |I| = |J|$ and consider the point

$$p_{I,J} := \frac{1}{k}\left(e_I + e_J\right).$$

Since $(I, J) \in \mathrm{IJ}_G$, we get that $p_{I,J} \in Q_G$, and therefore it must belong to Δ_T for some $T \in \mathcal{T}(\tau)$, in other words, there is a way to represent $p_{I,J}$ as a convex combination of vectors $e_i + e_{\bar{j}}$ for $(i, \bar{j}) \in T$. Let $F \subset T$ be the set of edges whose coefficients in this convex combination are nonzero. We claim that F is a partial matching with $\mathrm{I}(F) = I$ and $\mathrm{J}(F) = J$. Indeed, let $i \in [m]$ be a leaf of F adjacent to a single edge $(i, \bar{j}) \in F$. It follows that $i \in I$, $\bar{j} \in J$, and the coefficient of $(i, \bar{j})$ in the convex combination must be equal to $\frac{1}{k}$. Therefore $\bar{j}$ is not adjacent to any other edges of F. Since this holds for every leaf i of F (and similarly for every leaf $\bar{j}$ of F), we have shown that F is a partial matching, thus finishing the proof of part (5). □

Lemma 5.3 follows from our next observation.

Lemma 6.3. *Let τ be a triangulation of Q_G and $i \in [m]$. Consider a tree $T \in \mathcal{T}(\tau)$ and a forest $F \in \mathcal{RSM}(\tau)$ satisfying $\mathrm{LD}^-(T) + e_i = \mathrm{LD}(F)$. Then $N_i(F) = N_i(T)$.*

Proof. For each $i' \in [m]$ that is not equal to i, we have $\deg_{i'}(T) = \deg_{i'}(F) + 1$, so there exists a map $\bar{q} : [m] \setminus \{i\} \to [\bar{n}]$ satisfying $(i', \bar{q}(i')) \in T \setminus F$ for all $i' \in [m] \setminus \{i\}$.

Suppose that $N_i(F) \neq N_i(T)$. Then the map $\bar{q}$ can be extended to $[m]$ by setting $\bar{q}(i)$ to be any element of $N_i(T) \setminus N_i(F)$ (these two sets are of the same cardinality). After that, we have $(i', \bar{q}(i')) \in T \setminus F$ for all $i' \in [m]$. Thus the directed graph $U(T, F)$ contains a directed subgraph U' with edge set

$$\{i' \to \bar{q}(i') \mid i' \in [m]\} \cup \{\bar{j} \to i' \mid (i', \bar{j}) \in F\}. \tag{6.1}$$

By construction, U' has no directed cycles of length 2, and each vertex of this directed graph has outdegree 1. Thus U' contains a directed cycle, a contradiction. □

We now fix a triangulation τ of Q_G and proceed to showing that the map $\mathbb{T}_\tau$ satisfies the axioms of a trianguloid.

Lemma 6.4. *The map $\mathbb{T}_\tau$ satisfies Axioms (T1') and (T2').*

Proof. This is obvious from (5.1): $\mathbb{T}_\tau(\circ \underset{i}{\to} a)$ is equal to $N_i(F_\tau(a))$, so its cardinality is equal to the degree of i in $F_\tau(a)$, i.e., to a_i (by the definition of F_τ), which proves (T1'). Since $F_\tau(a)$ is a right semi-matching, for each $\bar{j} \in [\bar{n}]$ there exists a (unique) $i \in [m]$ such that $(i, \bar{j}) \in F_\tau(a)$, which proves (T2'). □

Lemma 6.5. *The map $\mathbb{T}_\tau$ satisfies Axiom (T3').*

Proof. Let a and $a' := a + e_i - e_j$ be two points of $P_G \cap \mathbb{Z}^m$, and let $F := F_\tau(a)$, $F' := F_\tau(a')$ be the corresponding elements of $\mathcal{RSM}(\tau)$. Consider the directed subgraph U' of $U(F, F')$ with all edges of $F \cap F'$ removed. We would like to show that $N_i(F) \subset N_i(F')$. Suppose that this is not the case, then clearly whenever a vertex of U' has an incoming edge, it also must have an outgoing edge. (This was already true for each vertex of U' except for possibly i.) Thus U' contains a directed cycle and we get a contradiction. □

We have thus shown that $\mathbb{T}_\tau$ is a pre-trianguloid, completing the proof of Proposition 5.2 as well as of its special case, Proposition 3.3. We finish by showing that $\mathbb{T}_\tau$ is in fact a trianguloid.

Lemma 6.6. *The map $\mathbb{T} := \mathbb{T}_\tau$ satisfies Axiom (T4').*

Proof. Let c, i, j, k be as in Axiom (T4'), and let $T := T_\tau(c + e_i) \in \mathcal{T}(\tau)$ be the tree with $\mathrm{LD}^-(T) = c + e_i$. Let $F := F_\tau(c + e_j + e_k) \in \mathcal{RSM}(\tau)$ be the forest with $\mathrm{LD}(F) = c + e_j + e_k$. Assume that $\mathbb{T}(c + e_i \underset{j}{\to} \bullet) \neq \mathbb{T}(c + e_k \underset{j}{\to} \bullet)$, which by Lemma 6.3 is equivalent to $N_j(T) \neq N_j(F)$, since $N_j(F) = \mathbb{T}(c + e_k \underset{j}{\to} \bullet)$ and $N_j(T) = \mathbb{T}(c + e_i \underset{j}{\to} \bullet)$. These two sets have the same cardinality $c_j + 1$, so there exists $\bar{q}(j) \in [\bar{n}]$ such that $(j, \bar{q}(j)) \in T \setminus F$. For each $i' \in [m] \setminus \{j, k\}$, we have $\deg_{i'}(T) > \deg_{i'}(F)$ and thus there exists $\bar{q}(i') \in [\bar{n}]$ such that $(i', \bar{q}(i')) \in T \setminus F$. Hence if $N_k(T) \neq N_k(F)$ then we can extend $\bar{q}$ to the whole $[m]$ and get a contradiction because $U(T, F)$ will contain a directed subgraph U' with edge set given by (6.1) that must have a cycle of length more than 2. We have shown that $N_k(T) = N_k(F)$, equivalently, $\mathbb{T}(c + e_i \underset{k}{\to} \bullet) = \mathbb{T}(c + e_j \underset{k}{\to} \bullet)$. The proof that $\mathbb{T}(c + e_j \underset{i}{\to} \bullet) = \mathbb{T}(c + e_k \underset{i}{\to} \bullet)$ is completely similar.

The only thing left to show is that $c+e_j$ belongs to P_G^-. It suffices to show that for any $t \in [m]$ we have $c+e_j+e_t \in P_G$. This is clear for $t=k$ so assume that $t \neq k$. We claim that there exists a sequence $(t_1, t_2, \dots, t_r)$ of distinct elements of $[m]$ such that $t_1 = t$, $t_r = k$, and for each $1 \leq s \leq r-1$, t_{s+1} is the unique vertex that is connected to $\bar{q}(t_s)$ in F. Indeed, we build such a sequence by induction: if $t_s \neq k$ then there exists a unique t_{s+1} connected to $\bar{q}(t_s)$ in F, and it must be different from $t_1, \dots, t_s$ since otherwise we would have found a directed cycle in $U(T,F)$. This process has to terminate, and since k is the only vertex for which $\bar{q}$ is undefined, we must have $t_r = k$ for some $r \geq 2$. Consider now the forest

$$F' = F \cup \{(t_s, \bar{q}(t_s)) \mid 1 \leq s \leq r-1\} \setminus \{(t_{s+1}, \bar{q}(t_s)) \mid 1 \leq s \leq r-1\}.$$

Clearly F' is a right semi-matching with $\mathrm{LD}(F') = c + e_j + e_t$ which implies that $c + e_j + e_t \in P_G$. We are done with the proof. □

7. From trianguloids to triangulations

We start by showing Propositions 5.4 and 3.4. The following lemma will be used many times throughout our proofs.

Lemma 7.1. *Let $\mathbb{T}$ be a pre-trianguloid, $b \in P_G^-$, and let $T := T_{\mathbb{T}}(b) \subset G$ be the subgraph given by* (3.1). *Suppose that for some $r \geq 1$ there exists a* simple path $\bar{j}_1, i_1, \bar{j}_2, \dots, i_{r-1}, \bar{j}_r, i_r$ *in T, i.e., there exist distinct indices $i_1, \dots, i_r \in [m]$ and $\bar{j}_1, \dots, \bar{j}_r \in [\bar{n}]$ such that for each $1 \leq s \leq r$ we have $(i_s, \bar{j}_s) \in T$ and for each $1 \leq s \leq r-1$ we have $(i_s, \bar{j}_{s+1}) \in T$. Then for all $1 \leq s \leq t \leq r$, we have*

$$\bar{j}_s \in \mathbb{T}(\circ \xrightarrow[i_s]{} b + e_{i_t}). \tag{7.1}$$

Proof. We note that for each $1 \leq t \leq r$, we have $b + e_{i_t} \in P_G$, so the edge $\circ \xrightarrow[i_s]{} b + e_{i_t}$ belongs to $E(\Gamma_G)$ for all $1 \leq s \leq t \leq r$. This includes the statement that the vector $b + e_{i_t} - e_{i_s}$ has nonnegative coordinates for all $1 \leq s \leq t \leq r$.

Fix $t \geq 1$. For $s = t$, the statement $\bar{j}_t \in \mathbb{T}(\circ \xrightarrow[i_t]{} b + e_{i_t})$ is by definition equivalent to $(i_t, \bar{j}_t) \in T$, which is the case for all $1 \leq t \leq r$. Let now $1 \leq s < t$ and suppose that (7.1) is proven for the pair $(s+1, t)$. By Axiom (T3'), $\mathbb{T}(\circ \xrightarrow[i_s]{} b + e_{i_t}) \subset \mathbb{T}(\circ \xrightarrow[i_s]{} b + e_{i_s})$, and by Axiom (T1'), their cardinalities satisfy

$$|\mathbb{T}(\circ \xrightarrow[i_s]{} b + e_{i_t})| + 1 = |\mathbb{T}(\circ \xrightarrow[i_s]{} b + e_{i_s})|.$$

Therefore there exists a unique index $\bar{j} \in \mathbb{T}(\circ \xrightarrow[i_s]{} b + e_{i_s}) \setminus \mathbb{T}(\circ \xrightarrow[i_s]{} b + e_{i_t})$. We claim that $\bar{j} = \bar{j}_{s+1}$. Indeed, we know that $\bar{j}_{s+1} \in \mathbb{T}(\circ \xrightarrow[i_s]{} b + e_{i_s})$ because $(i_s, \bar{j}_{s+1}) \in T$. On the other hand, by the induction hypothesis for $(s+1, t)$, we get that $\bar{j}_{s+1} \in \mathbb{T}(\circ \xrightarrow[i_{s+1}]{} b + e_{i_t})$. Therefore by Remark 3.2, $\bar{j}_{s+1} \notin \mathbb{T}(\circ \xrightarrow[i_s]{} b + e_{i_t})$ and we have shown that $\bar{j} = \bar{j}_{s+1}$. Since $\bar{j}_s \neq \bar{j}_{s+1}$ and $\bar{j}_s \in \mathbb{T}(\circ \xrightarrow[i_s]{} b + e_{i_s})$, it follows that $\bar{j}_s \in \mathbb{T}(\circ \xrightarrow[i_s]{} b + e_{i_t})$, which completes the proof. □

Remark 7.2. We will sometimes use a variant of Lemma 7.1 where the path starts with i_1 instead of $\bar{j}_1$ (but still ends with i_r). In this case, (7.1) applies to all $2 \le s \le t \le r$, with the same proof.

Proof of Propositions 5.4 and 3.4. Consider a vertex $b \in P_G^-$. We need to show that the collection $T := T_{\mathbb{T}}(b)$ of edges gives a spanning tree of G. By Axiom (T1'), T contains exactly $\sum_{i\in[m]}(b_i+1) = n+m-1$ edges. Thus we only need to show that it contains no cycles.

Suppose that T contains a cycle that consists of vertices $\bar{j}_1, i_1, \bar{j}_2, i_2 \dots, \bar{j}_r, i_r, \bar{j}_1$ in this order. By Lemma 7.1 applied to $s=1$ and $t=r$, we get that $\bar{j}_1 \in \mathbb{T}(\circ \underset{i_1}{\to} b+e_{i_r})$. On the other hand, $(i_r, \bar{j}_1) \in T$ so $\bar{j}_1 \in \mathbb{T}(\circ \underset{i_r}{\to} b+e_{i_r})$. This contradicts Remark 3.2. $\square$

Let us also use Lemma 7.1 to prove the following result which will be used in Section 8.

Lemma 7.3. *For a pre-trianguloid $\mathbb{T}$ and an index $\bar{j} \in [\bar{n}]$, consider the set*

$$P_G(\bar{j};\mathbb{T}) := \{a \in P_G \cap \mathbb{Z}^m \mid \exists\, b \in P_G^- :\ (b \to a) \in E(\Gamma_G) \text{ and } \bar{j} \in \mathbb{T}(b \to a)\}.$$

Then we have

$$P_G(\bar{j};\mathbb{T}) = \{b+e_i \mid b \in P_G^-,\ i \in N_{\bar{j}}(G)\}. \tag{7.2}$$

Note that the right hand side of (7.2) does not depend on $\mathbb{T}$.

Proof. First of all, it is clear that the left hand side of (7.2) is contained in the right hand side, since if b belongs to P_G^- and $\bar{j} \in \mathbb{T}(b \to a)$ then by Axiom (T2'), we have $a = b + e_i$ for some $i \in N_{\bar{j}}(G)$. Now assume that $a \in P_G \cap \mathbb{Z}^m$ does not belong to the left hand side of (7.2). Let i be the (unique by Remark 3.2) index such that $\bar{j} \in \mathbb{T}(\circ \underset{i}{\to} a)$. Then our assumption implies $a - e_i \notin P_G^-$. If a does not belong to the right hand side of (7.2) then we are done. Otherwise let $k \in N_{\bar{j}}(G)$ be an index such that $a - e_k \in P_G^-$. Consider the tree $T := T_{\mathbb{T}}(a-e_k)$, and let $\bar{j}_1, i_1, \bar{j}_2, \dots, i_r$ be the path in T that connects $\bar{j} = \bar{j}_1$ to $k = i_r$. By Lemma 7.1 applied to $s=1$ and $t=r$, we get $\bar{j} \in \mathbb{T}(\circ \underset{i_1}{\to} a)$. By Remark 3.2, this implies that $i_1 = i$. Therefore

$$T' := (T \setminus \{(i,\bar{j})\}) \cup \{(k,\bar{j})\}$$

is again a tree, and since it satisfies $\mathrm{LD}^-(T') = a - e_i$, we get that $a - e_i \in P_G^-$, which contradicts our assumption. We are done with the proof. $\square$

Finally, we focus on proving Theorems 5.6 and 3.6.

Lemma 7.4. *Let $\mathbb{T}$ be a pre-trianguloid and consider two trees $T := T_{\mathbb{T}}(c+e_i)$, $T' := T_{\mathbb{T}}(c+e_k)$ for some $i \neq k \in [m]$ and c such that $c+e_i, c+e_k \in P_G^-$. Suppose in addition that $|T \cap T'| = m+n-2$. Then T and T' are compatible.*

Proof. Let $F := T \cap T'$ and suppose that T and T' are not compatible. This is equivalent to saying that i and k belong to the same connected component of F, so consider a path $i_1, \bar{j}_2, i_2, \bar{j}_3, \dots, \bar{j}_r, i_r$ in F such that $i_1 = i$ and $i_r = k$ (here $r \ge 2$). Applying Lemma 7.1 to $b = c+e_i$, $s=2$, and $t=r$ shows $\bar{j}_2 \in \mathbb{T}(\circ \underset{i_2}{\to} c+e_i+e_k)$. On the other hand, the edge $(i, \bar{j}_2)$ belongs to $T' = T_{\mathbb{T}}(c+e_k)$ and thus $\bar{j}_2 \in \mathbb{T}(\circ \underset{i}{\to} c+e_i+e_k)$. This contradicts Remark 3.2 since we have assumed $i \neq i_2$. $\square$

Proposition 7.5. *Let $T \subset G$ be a spanning tree of G and consider an edge $(v, \bar{u}) \in T$. Define $F := T \setminus \{(v, \bar{u})\}$. Then the following conditions are equivalent:*

- (i) *The simplex Δ_F is not contained inside the boundary of Q_G.*
- (ii) *There exists an edge $(v', \bar{u}') \in G$ such that v' (resp., $\bar{u}'$) belongs to the connected component of F that contains $\bar{u}$ (resp., v).*

If (i) or (ii) holds then we call $(v, \bar{u})$ a *replaceable edge* of T.

Proof. To show that (ii) implies (i), observe that the trees T and $T' := F \cup \{(v', \bar{u}')\}$ are compatible, and thus the corresponding top-dimensional simplices $\Delta_T, \Delta_{T'} \subset Q_G$ intersect by Δ_F. Therefore the relative interior of Δ_F is contained inside the relative interior of Q_G. Conversely, suppose that there is no edge $(v', \bar{u}')$ satisfying (ii). Define $I \subset [m]$, $J \subset [\bar{n}]$ so that $I \cup J$ is the connected component of F containing v. It follows that there are no edges in G between J and $[m] \setminus I$, and that there are no edges between I and $[\bar{n}] \setminus J$ in F. Consider a linear function $h : \mathbb{R}^{m+n} \to \mathbb{R}$ defined by

$$h(x_1, \dots, x_m, x_{\bar{1}}, \dots, x_{\bar{n}}) := \sum_{i \in [m] \setminus I} x_i + \sum_{\bar{j} \in J} x_{\bar{j}}.$$

Since there are no edges in G between J and $[m] \setminus I$, the value of h on $e_i + e_{\bar{j}}$ for $(i, \bar{j}) \in G$ is at most 1. Thus the maximum value of h on Q_G is 1. On the other hand, for every edge $(i, \bar{j}) \in F$ we have either $i \in [m] \setminus I$ or $\bar{j} \in J$. Thus h is identically equal to 1 on Δ_F. Since T contains the edge $(v, \bar{u})$ and $h(e_v + e_{\bar{u}}) = 0$, we get that the maximum of h is attained at a facet of Q_G that contains Δ_F, finishing the proof of the proposition. $\square$

Lemma 7.6. *Let $\mathbb{T}$ be a trianguloid and consider a point $b \in P_G^-$. Then for any replaceable edge $(v, \bar{u})$ of $T_{\mathbb{T}}(b)$, there exists a point $b' \in P_G^-$ such that $T_{\mathbb{T}}(b) \setminus T_{\mathbb{T}}(b') = \{(v, \bar{u})\}$.*

Proof. Denote $c := b - e_v$, so $(v, \bar{u})$ is a replaceable edge of $T_{\mathbb{T}}(c + e_v)$. Let

$$B := \{i \in [m] : \ c + e_i \in P_G^-\},$$

and consider a subset $B' \subset B$ consisting of all indices $i \in B$ such that

$$\bar{u} \notin \mathbb{T}(c + e_i \underset{v}{\to} \bullet). \tag{7.3}$$

Note that $v \in B$ and $v \notin B'$. We claim that the set B' is non-empty. Indeed, let $(v', \bar{u}') \in G$ be any edge satisfying the conditions of Proposition 7.5, part (ii). Our goal is to prove that v' belongs to B'. First note that $c + e_{v'} \in P_G^-$ since replacing $(v, \bar{u})$ with $(v', \bar{u}')$ in T produces a spanning tree T' of G with $\mathrm{LD}^-(T') = c + e_{v'}$. We now need to show that (7.3) holds for $i := v'$.

Consider the path $i_1, \bar{j}_2, i_2, \dots, \bar{j}_r, i_r$ from $i_1 := v$ to $i_r := v'$ in T. It must pass through the edge $(v, \bar{u})$, thus $\bar{j}_2 = \bar{u}$. Applying Lemma 7.1 to $b = c + e_v$, $s = 2$ and $t = r$ yields

$$\bar{u} \in \mathbb{T}(\circ \underset{i_2}{\to} c + e_v + e_{v'}).$$

By Remark 3.2, we therefore have

$$\bar{u} \notin \mathbb{T}(\circ \underset{v}{\to} c + e_v + e_{v'}).$$

Thus indeed (7.3) holds for $i := v'$, and we have shown that $B' \neq \emptyset$.

Recall that $v \notin B'$. For $i \in B'$ and $j \in [m] \setminus \{v\}$, define

$$M(i,j) = \begin{cases} 1, & \text{if } j = i \text{ or } \mathbb{T}(c + e_v \underset{j}{\to} \bullet) = \mathbb{T}(c + e_i \underset{j}{\to} \bullet); \\ 0, & \text{otherwise.} \end{cases}$$

Our main goal is to find $i \in B'$ such that $M(i,j) = 1$ for all $j \neq v$. Indeed, for such i we clearly have $T_{\mathbb{T}}(c + e_v) \setminus T_{\mathbb{T}}(c + e_i) = \{(v, \bar{u})\}$. We first show that for any $i \in B'$ and $j \notin B'$ such that $j \neq v$ we have $M(i,j) = 1$. Indeed, suppose that $j \notin B'$. If $j \notin B$ then $c + e_j \notin P_G^-$ so by Axiom (T4'), we must have $M(i,j) = 1$. If $j \in B \setminus B'$ then we have $\bar{u} \in \mathbb{T}(c + e_j \underset{v}{\to} \bullet)$ but $\bar{u} \notin \mathbb{T}(c + e_i \underset{v}{\to} \bullet)$. Thus $\mathbb{T}(c + e_j \underset{v}{\to} \bullet) \neq \mathbb{T}(c + e_i \underset{v}{\to} \bullet)$. Applying Axiom (T4') to these three indices, we immediately get $M(i,j) = 1$. We have shown that $M(i,j) = 1$ for all $i \in B'$ and $j \notin B' \cup \{v\}$. It remains to find $i \in B'$ such that for all $j \in B'$ we have $M(i,j) = 1$.

Axiom (T4') imposes certain restrictions on $M(i,j)$. First, applying it to distinct indices v, i, j, we get

$$M(i,j) + M(j,i) > 0 \quad \text{for all } i, j \in B'. \tag{7.4}$$

Second, applying it to distinct indices $i, j, k \in B'$ yields the following:

$$\text{if } M(i,k) \neq M(j,k) \text{ then } M(k,i) = M(j,i) \text{ and } M(k,j) = M(i,j). \tag{7.5}$$

Indeed, $M(i,k) \neq M(j,k)$ implies that $\mathbb{T}(c + e_i \underset{k}{\to} \bullet) \neq \mathbb{T}(c + e_j \underset{k}{\to} \bullet)$, because one of these two sets is equal to $\mathbb{T}(c + e_v \underset{k}{\to} \bullet)$ while the other one is not. Applying Axiom (T4'), we get that $\mathbb{T}(c + e_k \underset{i}{\to} \bullet) = \mathbb{T}(c + e_j \underset{i}{\to} \bullet)$ and $\mathbb{T}(c + e_k \underset{j}{\to} \bullet) = \mathbb{T}(c + e_i \underset{j}{\to} \bullet)$. These conditions imply that $M(k,i) = M(j,i)$ and $M(k,j) = M(i,j)$, respectively.

We now prove that any $r \times r$ matrix $M(i,j)$ satisfying $M(i,i) = 1$ for all i together with (7.4) and (7.5), has a row filled with ones. We do this by induction on the size $r = |B'|$ of M. We have shown that B' is non-empty, so the base case is $r = 1$ which is clear. Suppose that $r > 1$ and by induction we may assume that there exists $1 \leq i < r$ such that $M(i,j) = 1$ for all $1 \leq j < r$. If $M(i,r) = 1$ then we are done, so suppose that $M(i,r) = 0$. We are going to show that in this case, $M(r,j) = 1$ for all $1 \leq j \leq r$. First, by (7.4), $M(r,i) = 1$. We also know that $M(r,r) = 1$. Now consider $1 \leq j < r$ such that $j \neq i$. If $M(j,r) = 0$ then by (7.4), $M(r,j) = 1$ and we are done. So suppose that $M(j,r) = 1$. Then $M(j,r) \neq M(i,r)$, so applying (7.5) yields $M(r,j) = M(i,j)$, and by the induction hypothesis, $M(i,j) = 1$. We have shown that $M(r,j) = 1$ for all $1 \leq j \leq r$, thus finishing the induction step together with the proof of the lemma. □

We are now ready to prove our main result.

Proof of Theorem 5.6. Note that we have already shown in Section 6 that if τ is a triangulation of Q_G then $\mathbb{T}_\tau$ defined by (5.1) is a trianguloid. Suppose now that $\mathbb{T}$ is a trianguloid, and let $\mathcal{T}(\mathbb{T}) = \{T_{\mathbb{T}}(b) \mid b \in P_G^- \cap \mathbb{Z}^m\}$ be the corresponding collection of trees. We would like to show that the simplicial complex $\tau_{\mathbb{T}}$ whose top-dimensional simplices are $\{\Delta_T \mid \mathcal{T}(\mathbb{T})\}$ is a triangulation of Q_G.

So far we have the following situation:

(1) $\{\Delta_T \mid \mathcal{T}(\mathbb{T})\}$ is a collection of top-dimensional simplices inside Q_G, each of them has the same volume, and their total volume equals the volume of Q_G;
(2) for every $T \in \mathcal{T}(\mathbb{T})$, if a facet Δ_F of Δ_T is not contained inside the boundary of Q_G then there exists $T' \in \mathcal{T}(\mathbb{T})$ such that $\Delta_T \cap \Delta_{T'} = \Delta_F$.

Indeed, as we have already noted, Claim (1) is explained in [Pos09, Lemma 12.5]. Claim (2) is proven as follows. Let $(v, \bar{u})$ be the unique edge in $T \setminus F$. Note that Δ_F not being contained inside the boundary of Q_G by Proposition 7.5 implies that $(v, \bar{u})$ is a replaceable edge in T. Then by Lemma 7.6, there exists another tree $T' \in \mathcal{T}(\mathbb{T})$ such that $T \setminus T' = \{(v, \bar{u})\}$. Finally, by Lemma 7.4, the trees T and T' are compatible, and thus $\Delta_T \cap \Delta_{T'} = \Delta_F$.

We need to prove two claims:

(a) The set $R_{\mathbb{T}} := \cup_{T \in \mathcal{T}(\mathbb{T})} \Delta_T$ equals Q_G.
(b) For $T, T' \in \mathcal{T}(\mathbb{T})$, we have $\Delta_T \cap \Delta_{T'} = \Delta_{T \cap T'}$.

To show (a), choose any point $q \in Q_G$ and suppose that it does not belong to $R_{\mathbb{T}}$. Choose a generic point $r \in R_{\mathbb{T}}$ and find the smallest $0 < t < 1$ such that $p := (1-t)q + tr \in R_{\mathbb{T}}$. Since r is generic, p must belong to a facet Δ_F of Δ_T for some $T \in \mathcal{T}(\mathbb{T})$. This facet is clearly not contained inside the boundary of Q_G, thus there exists another tree $T' \in \mathcal{T}(\mathbb{T})$ such that $F \subset T'$ and $\Delta_T \cap \Delta'_T = \Delta_F$. We get a contradiction with the minimality of t, thus finishing the proof of (a). Since the total volume of the simplices in $\tau_{\mathbb{T}}$ equals the volume of Q_G, it follows that any two simplices in $\tau_{\mathbb{T}}$ have disjoint interiors.

We now prove (b). Let F be any forest of G such that $F \subset T$ for some $T \in \mathcal{T}(\mathbb{T})$ and let $\mathcal{T}_F(\mathbb{T}) = \{T \in \mathcal{T}(\mathbb{T}) \mid F \subset T\}$. Choose any point f in the relative interior of Δ_F. By the same argument as in the previous paragraph, there exists a small ball B around f that is fully contained inside $R_F := \cup_{T \in \mathcal{T}_F(\mathbb{T})} \Delta_T$. (Indeed, we just choose B to be such that for all $T \in \mathcal{T}_F(\mathbb{T})$ and any facet of Δ_T that does not contain f, B does not intersect the hyperplane containing this facet.)

Thus for any $T \in \mathcal{T}(\mathbb{T}) \setminus \mathcal{T}_F(\mathbb{T})$, Δ_T cannot contain f because then its interior will intersect B and thus it will also intersect the interior of $\Delta_{T'}$ for some $T' \in \mathcal{T}_F(\mathbb{T})$. We have shown (b) which finishes the proof of Theorems 5.6 and 3.6. Note also that the map $\mathbb{T} \mapsto \tau_{\mathbb{T}}$ is inverse to the map $\tau \mapsto \mathbb{T}_\tau$ by Lemma 5.3. $\square$

8. Proof of Theorems 3.7 and 5.7

Suppose that τ and τ' are two different triangulations of Q_G and let $\mathbb{T} := \mathbb{T}_\tau$, $\mathbb{T}' := \mathbb{T}_{\tau'}$ be the corresponding trianguloids. We are going to show that the maps $\phi_\tau, \phi_{\tau'} : P_G^- \cap \mathbb{Z}^m \to P_{G^*}^- \cap \mathbb{Z}^n$ must be different.

For the sake of contradiction, assume that the maps ϕ_τ and $\phi_{\tau'}$ are the same. Thus for each $b \in P_G^- \cap \mathbb{Z}^m$ and each $\bar{u} \in [\bar{n}]$, the degree of $\bar{u}$ in $T_{\mathbb{T}}(b)$ equals the degree of $\bar{u}$ in $T_{\mathbb{T}'}(b)$. However, since the triangulations themselves are different, there exists $b \in P_G^- \cap \mathbb{Z}^m$ and $\bar{u} \in [\bar{n}]$ such that $N_{\bar{u}}(T_{\mathbb{T}}(b)) \neq N_{\bar{u}}(T_{\mathbb{T}'}(b))$. Let us fix this $\bar{u}$ and show that for some $b \in P_G^- \cap \mathbb{Z}^m$, the degrees of $\bar{u}$ in $T_{\mathbb{T}}(b)$ and $T_{\mathbb{T}'}(b)$ must be different. We introduce the following two subsets of $E(\Gamma_G)$:

$$A := \{(b \to a) \in E(\Gamma_G) \mid a \in P_G \cap \mathbb{Z}^m,\ b \in P_G^- \cap \mathbb{Z}^m : \bar{u} \in \mathbb{T}(b \to a)\};$$
$$A' := \{(b \to a) \in E(\Gamma_G) \mid a \in P_G \cap \mathbb{Z}^m,\ b \in P_G^- \cap \mathbb{Z}^m : \bar{u} \in \mathbb{T}'(b \to a)\}.$$

Lemma 8.1. *For every point $a \in P_G$, exactly one of the following is true:*

- *a is incident to exactly edge in A and to exactly one edge in A';*
- *a is not incident to any edge in $A \cup A'$.*

Proof. By Remark 3.2 together with Lemma 7.3, for each $a \in P_G(\bar{j}; \mathbb{T}) = P_G(\bar{j}, \mathbb{T}')$, there is exactly one $b \in P_G^-$ (resp., $b' \in P_G^-$) such that $(b \to a) \in A$ (resp., $(b' \to a) \in A'$). For $a \notin P_G(\bar{j}; \mathbb{T})$, there is no such b (resp., b'). □

For each $b \in P_G^- \cap \mathbb{Z}^m$, define $N(b) := N_{\bar{u}}(T_{\mathbb{T}}(b))$ and $N'(b) := N_{\bar{u}}(T_{\mathbb{T}'}(b))$. Thus for $b \in P_G^- \cap \mathbb{Z}^m$, we have

$$N(b) = \{i \in [m] \mid (b \underset{i}{\to} \bullet) \in A\}, \quad N'(b) = \{i \in [m] \mid (b \underset{i}{\to} \bullet) \in A'\}.$$

We claim that if $|N(b)| = |N'(b)|$ for all $b \in P_G^- \cap \mathbb{Z}^m$ then $A = A'$, or equivalently, $N(b) = N'(b)$ for all $b \in P_G^- \cap \mathbb{Z}^m$.

Let us denote $B := A \setminus A'$ and $B' := A' \setminus A$.

Lemma 8.2. *Suppose that for some $c \in \mathbb{Z}^m$ and distinct indices $i, j, k \in [m]$, we have $c + e_i, c + e_j \in P_G^-$. Assume in addition that*

$$(c + e_i \underset{k}{\to} \bullet) \in B, \quad (c + e_i \underset{j}{\to} \bullet) \in B', \quad (c + e_j \underset{i}{\to} \bullet) \in B.$$

Then $(c + e_j \underset{k}{\to} \bullet) \in B$.

Proof. If $c + e_k \notin P_G^-$ then by Axiom (T4'), we have $\mathbb{T}(c + e_i \underset{k}{\to} \bullet) = \mathbb{T}(c + e_j \underset{k}{\to} \bullet)$ and $\mathbb{T}'(c + e_i \underset{k}{\to} \bullet) = \mathbb{T}'(c + e_j \underset{k}{\to} \bullet)$ so if $(c + e_i \underset{k}{\to} \bullet) \in B$ then the same holds for $(c + e_j \underset{k}{\to} \bullet)$. Suppose now that $c + e_k \in P_G^-$. Consider the point $c + e_i + e_k \in P_G$. We know that $(c + e_i \underset{k}{\to} \bullet) \in B$, and thus by Remark 3.2, $(c + e_k \underset{i}{\to} \bullet) \notin A$. Therefore $\mathbb{T}(c + e_j \underset{i}{\to} \bullet) \neq \mathbb{T}(c + e_k \underset{i}{\to} \bullet)$, so by Axiom (T4') we have $\mathbb{T}(c + e_i \underset{k}{\to} \bullet) = \mathbb{T}(c + e_j \underset{k}{\to} \bullet)$, and therefore $(c + e_j \underset{k}{\to} \bullet) \in A$. The only thing left to show is that $(c + e_j \underset{k}{\to} \bullet) \notin A'$. Indeed, suppose otherwise that $(c + e_j \underset{k}{\to} \bullet) \in A'$. By Remark 3.2, we get that $(c + e_k \underset{j}{\to} \bullet) \notin A'$ which implies $\mathbb{T}'(c + e_i \underset{j}{\to} \bullet) \neq \mathbb{T}'(c + e_k \underset{j}{\to} \bullet)$. On the other hand, we also have $\mathbb{T}'(c + e_i \underset{k}{\to} \bullet) \neq \mathbb{T}'(c + e_j \underset{k}{\to} \bullet)$ because we know that $(c + e_i \underset{k}{\to} \bullet) \notin A'$. These two conditions together violate Axiom (T4'), and thus $(c + e_j \underset{k}{\to} \bullet) \notin A'$. We are done with the proof. □

For $b \in P_G^- \cap \mathbb{Z}^m$, denote $M(b) := N(b) \setminus N'(b)$ and $M'(b) = N'(b) \setminus N(b)$, thus we have $|M(b)| = |M'(b)|$ for all b. Let us find the point $b^{(1)} \in P_G^- \cap \mathbb{Z}^m$ for which $|M(b^{(1)})| = |M'(b^{(1)})|$ is maximal. Choose some $i \in M(b^{(1)})$, thus $(b^{(1)} \underset{i}{\to} \bullet) \in B$. We are going to construct an infinite sequence of points $b^{(1)}, b^{(2)}, \dots$ such that for all $s \geq 1$, $M(b^{(s)}) = M(b^{(1)})$, $M'(b^{(s)}) = M'(b^{(1)})$, and $b^{(s+1)} = b^{(s)} + e_i - e_{j^{(s)}}$ for some $j^{(s)} \in [m] \setminus \{i\}$. Clearly this leads to a contradiction because the i-th coordinate of a point in P_G^- cannot be arbitrarily large.

Let us show how to construct the point $b^{(2)}$ from $b^{(1)}$. By Lemma 8.1, there exists a unique index $j^{(1)} \in [m]$ such that $(\circ \xrightarrow[j^{(1)}]{} b^{(1)} + e_i) \in B'$, so denote $b^{(2)} := b^{(1)} + e_i - e_{j^{(1)}}$. By Lemma 8.1 again, we have $b^{(2)} \in P_G^- \cap \mathbb{Z}^m$.

We claim that $M(b^{(1)}) = M(b^{(2)})$ and $M'(b^{(1)}) = M'(b^{(2)})$. Indeed, let $k \neq i, j^{(1)}$ be any element of $M'(b^{(1)})$. Letting $c := b^{(1)} - e_{j^{(1)}}$, we have

$$(c + e_{j^{(1)}} \underset{k}{\to} \bullet) \in B', \quad (c + e_{j^{(1)}} \underset{i}{\to} \bullet) \in B, \quad (c + e_i \xrightarrow[j^{(1)}]{} \bullet) \in B'.$$

Then by Lemma 8.2 (with B and B' swapped), we must have $(c + e_i \underset{k}{\to} \bullet) \in B'$, and thus $k \in M'(b^{(2)})$. In addition, we know that $i \notin M'(b^{(1)})$ and $j^{(1)} \in M'(b^{(2)})$. It follows that $M'(b^{(1)}) \subset M'(b^{(2)})$, but since $M'(b^{(1)})$ has maximal size, we must have $M'(b^{(1)}) = M'(b^{(2)})$. Switching the roles of $b^{(1)}$ and $b^{(2)}$ and of i and $j^{(1)}$, we see that $M(b^{(1)}) = M(b^{(2)})$. In particular, we have $i \in M(b^{(2)})$. Repeating this argument for $b^{(2)}$, we find $b^{(3)}$, etc. As we have noted earlier, constructing such an infinite sequence leads to a contradiction with the assumption that $M(b)$ is non-empty. This finishes the proof of Theorems 5.7 and 3.7. □

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